AN INFORMAL INTRODUCTION TO SOME FORMAL CONCEPTS FROM LEWIN'S TRANSFORMATIONAL THEORY[1]

Ramon Satyendra

Since the publication of David Lewin's *Generalized Musical Intervals and Transformations* (GMIT),[2] transformational theory has become an active area within the discipline of music theory. Though this subdiscipline has been with us for some years now, a dialogue between transformational specialists and others has been slow to develop. The emergence of broad-based critique and commentary has been inhibited in part perhaps by the language barrier posed by the theory's mathematics. This article aims to bridge the gap by exploring in an informal context some of the mathematical ideas from GMIT out of which later work has grown.[3]

GMIT's generalized notion of interval encompasses both "Cartesian" and "transformational" ways of conceptualizing music. How this double view finds expression in the particulars of the theory is a central theme of this article. Part one of the article considers what is entailed in this philosophical dichotomy and how it is dissolved by the generalizing power of the theory. As will be discussed, a musical system may be formed by combining a group of musical transformations with a set of musical objects. When the group and set are combined in the right way the result is a "simply transitive group action on a set," a mathematical construct brought into GIS theory to represent a transformational perspective. Parts two and three introduce three types of mappings including the interval function of a GIS. A knowledge of these is a prerequisite to the study of GISs and neo-Riemannian groups. Part four explores the Cartesian perspective offered by the definition of a GIS and demonstrates the kinship between Cartesian and transformational perspectives by constructing a GIS from a simply-transitive-group system. Part five outlines relationships between atonal theory's system of transpositions and inversions, neo-Riemannian triadic systems, and Lewin's simply-transitive-group systems.[4] The last part of the article considers "non-commutative" GISs. Two of Lewin's innovative findings will be illustrated. First, given any non-commutative GIS, one can always discover another system, its *dual*, which exists in a sort of "parallel universe" of transformations. Second, in any non-commutative GIS, transposing a pair of notes by the same amount can actually change the interval between the notes! In this circumstance, there always exist operations (which are not transpositions) that preserve intervals. Non-commutative GISs, dual GISs, and non-transposing, interval-preserving operations are especially original topics in GMIT not treated elsewhere.

I. A Transformational Perspective.

In GMIT, Lewin presents the idea of "interval" first in terms of a *generalized interval system (GIS)*, and then in terms of a *simply transitive group action on a set*. After both presentations, he remarks that the first perspective is Cartesian and observer-oriented whereas the second perspective is gestural and subject-oriented. He also notes that the second is more general: while a generalized interval system captures our historically grounded idea of interval as distance or measurement, a simply transitive group captures *both* the idea of interval as distance and of interval as transformation (GMIT 158–59). We begin by looking at the transformational side of the dichotomy, the simply-transitive-group (STRANS) perspective.

What is meant by the word "system" in "generalized interval system" or "STRANS system"? Some may fear that a systematic procedure for analyzing music is being offered. Actually, this is not at all the case. In GMIT, "system" is being used in the dictionary sense of "an interdependent group of items forming a unified whole." The definition of a simply transitive group (Definition 1) refers to a system composed of two items: the "space," denoted by "S," and the "group" whose elements transform elements of the space. A group and a space together form a system which mathematicians call a "simply transitive group action on a set" if they jointly fulfill the simply transitivity condition specified in Definition 1.[5]

Definition 1. The group STRANS of operations on S is simply transitive when the following condition is satisfied: Given any elements s and t of S, then there exists a unique member OP of STRANS such that OP(s) = t. (GMIT 157).

In our first pass over the definition in Definition 1 we will focus on the terms "S" ("S" stands for musical *space*) and "group" used in the definition. The first thing to note is both terms denote sets.[6] Atonal theory defines a *set* to be a collection of objects and this informal definition will serve us also in GIS theory. While "space" and "group" each refer to a set, the objects in each are different in kind. It is useful to liken the difference between space-elements and group-elements to the difference between nouns and verbs. Group elements may be likened to verbs referring to musical actions such as "to transpose," "to invert," "to transform," whereas space-elements may be likened to nouns referring to musical objects such as "notes," "durations," "triads." A space contains the musical objects upon which actions are performed; a group contains the actions themselves.

In Figure 1, G is the familiar T/I group of atonal theory consisting of the twelve transposition and twelve inversion operations. When listening to an octatonic piece (such as Scriabin, op. 74, no. 3) one might notice the octatonic collection on one hand and the processes of transposition and inversion on the other. G and S in Figure 1 together form a musical system that accords with these two aspects of our listening experience. The pitches comprise the space S and the actions performed on those pitches comprise the group G.[7]

Is Figure 1 an STRANS system? It shows a group G of actions and a musical space S of objects, so we have two of the three things we need. But Definition 1 also asks that a simple transitivity condition be satisfied. The condition is this: for any two space-elements (say the pitch classes C and D♯) there must be exactly one group-element that will transform the first into the second. Is this true for our musical system? The answer is no. There are not one but two group elements that transform C into D♯: group-element T_3 and I_3 both take C to D♯. So our proposed musical system is not an STRANS system.

The existence here of two answers is familiar; in the T/I group there are always two ways to get from one note to another. We can get from F

G = "T/I group" = $\{T_0, T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8, T_9, T_{10}, T_{11}, I_0, I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, I_9, I_{10}, I_{11}\}$

S = {C, C♯, D♯, E, F♯, G, A, B♭}.

Figure 1

to G by T_2 or by I_0, we can get from B♭ to F♯ by T_8 or by I_4. From this we see that *the familiar atonal system of transposing and inverting pitch classes is not an STRANS system.* What is gained by the unique-way requirement of an STRANS system? After all, the familiar atonal system seems to work well without fulfilling this requirement. We will return to this important question later. But for now let us adjust the system in Figure 1 to fit the definition of an STRANS system.

We now select a smaller group, $G' = \{T_0, T_3, T_6, T_9, I_1, I_4, I_7, I_{10}\}$ while keeping the same elements of S. Does this modified system satisfy the simple transitivity condition? Now the only group element that takes C to D♯ is T_3. Let us try a few more examples. I_1 is the unique element in G′ that takes E to A, T_6 is the unique element that takes E♭ to A, I_{10} is the unique element that takes C to A♯. It can be verified that for any pair of pitches in S, there is a unique group element in G′ that takes the first to the second. So this modified system (S and G′) involves a simply transitive group action on the set S—it conforms to the definition of STRANS.

We have suggested that elements of a group are actions, principles of movement. This verb-oriented attitude corresponds to the STRANS perspective. The GIS, Cartesian perspective calls on us to see the selfsame group elements differently, as nouns of distance or measurement. Both perspectives are two sides of the same coin and there is an advantage to seeing musical situations in both ways. Sentences (1)–(6) illustrate this unified perspective by paraphrasing ordinary statements about music. Each statement appears in two versions, one which emphasizes the verb of action, the other the noun of distance. When the emphasis is on the verb ((1), (3), and (5)), the transformational perspective seems dominant and when the emphasis is on the noun ((2), (4), (6)) the Cartesian perspective seems dominant.

(1) In the recapitulation, Haydn **transposes** the second theme to the tonic key. [Transformational]

(2) The distance between the two instances of the second theme is a **fifth**. [Cartesian.]

(3) The first hexachord **inverted and then transposed by seven** gives you the second hexachord. [Transformational.]

(4) The distance between the first and second hexachord is $\mathbf{T_7I}$. [Cartesian.]

(5) The music in C major **modulates** to A♭ major. [Transformational.]

(6) The **distance** between the keys of C major and A♭ major is greater than the distance between the keys of C major and G major. [Cartesian.]

It is immediately evident that some sentences are more idiomatic in one form over another. It may seem odd in (4) to speak of the "distance" between hexachords as T_7I. However, such speaking is actually like describing the distance between keys in (6). Though we are accustomed to thinking of particular musical situations in one way, it is possible to move between perspectives. Lewin notes there is a payoff that comes from embracing a generalized conception which captures both senses of "interval" (GMIT 159).

> [A]bove we sketched a mathematical dichotomy between intervals in a GIS and transposition-operations on a space: Either can be generated formally from the characteristic properties of the other. More significant than this dichotomy, I believe, is the *generalizing* power of the transformational attitude. It enables us to *subsume* the theory of GIS structure, along with the theory of simply transitive groups, into a broader theory of transformations. This enables us to consider intervals-between-things and transpositional-relations-between-Gestalts not as alternatives, but as the *same* phenomenon manifested in different ways.

With this generalized notion of interval in mind we can answer our earlier question about the significance of the simple transitivity condition. When reckoning intervallic distances we intuitively expect unique answers. It is counterintuitive to describe the straight-line distance between the chair and the table as *both* two feet and three feet. By requiring that a musical system satisfy the simple transitivity condition we are assured that the interval formed between any two points in a musical space may be uniquely determined. If a system is not simply transitive, it becomes counterintuitive to shift between transformational and intervallic perspectives. For instance it is intuitive to say that both T_3 and I_3 transform C to E♭, but it is counterintuitive to think of the interval between C and E♭ as *both* T_3 and I_3.[8]

More can be said on the systemic meaning of simple transitivity, but from just the foregoing we can appreciate how Lewin's theory and atonal theory differ in orientation. In atonal theory, one does not typically ask that a twelve-tone operation (such as I_4) do double duty as a kind of interval measurement. Consequently, in atonal theory a concern about simple transitivity—that is about the uniqueness of interval measurements—does not arise. However, if one is thinking of transformations (including twelve-tone operations) as intervals, with the traditional connotation of intervals as Cartesian extension, then the issue of simple transitivity arises immediately. The simple transitivity condition is what GIS theory needs to permit us to go back and forth conceptually between a view of intervals as transformation and of intervals as Cartesian extension.

II. Mappings in Transformational Theory.

Consider the sets S and S′: S = {C, C♯, D, E♭, E, F}, and S′ = {pp, p, mp, mf, f, ff}. Figure 2a defines a mapping between them via a two-column table. In a mapping table the order of the rows does not matter (though the order of items within each row does). For example, C♯ in Figure 2a is associated with p regardless of the row order. Mappings between unlike sets ($S \rightarrow S'$ where $S \neq S'$) arise in cross-domain analyses.[9] Note that in Figure 2a exactly one element in S′ is assigned to each element in S. This uniqueness is the defining feature of a *mapping*: *a mapping f from a set S to another set S′ (notated as f: $S \rightarrow S'$) is a rule that assigns exactly one element in S′ for each element of S.* (GMIT 1).

Mapping is synonymous with the mathematical term *function.* Mappings are often notated with parentheses in expressions such as f(x) = y, T_4(G) = B, and I_5(C) = F.[10] The first expression can be rewritten with arrows in the form f: $x \rightarrow y$, which is read "f maps x to y, or "the rule f assigns y to x." The latter expressions can be rewritten as T_4: $G \rightarrow B$, I_5: $C \rightarrow F$. When referring to a mapping between sets one writes expressions such as $X \rightarrow Y$ or $S \rightarrow S'$ with upper-case letters.

Case 1, f: $S \rightarrow S$. Transformations and Operations. Though it is possible to have a mapping between unlike sets as in Figure 2a, GMIT reserves the terms "transformation" and "operation" to mappings between *like* sets, mappings of the form f: $S \rightarrow S$. *A transformation is formally defined as a mapping from a set to itself. An operation is the special case of a transformation which is "one-to-one and onto"* (GMIT 3). The meaning of "one-to-one and onto" is illustrated in Figure 2b, a mapping table for the group element T_3.

We see that T_3 maps elements of S to itself, where S is the 12 pitch-classes, hence T_3 is a transformation. The two defining traits of an operation are visible in Figure 2b: (1) Every element of S appears on the receiving side of the arrows. This means the mapping is *onto*. (2) Every element of S appears precisely once in each column. This means the mapping is *one-to-one*. Since this mapping is one-to-one and onto and from a set to itself, we conclude that T_3 is an *operation*. Instead of "operation" some authors use the synonymous term *permutation*.

Figure 2c illustrates a transformation called "wedging-to-E" which Lewin uses in his analysis of "Angst und Hoffen," Schoenberg op. 15, no. 7. The transformation is called wedging-to-E because it moves each note towards E by the shortest "clockface" distance (reckoning from a clock whose hours are the pitch classes of the chromatic arranged in a clockwise "ascending" sequence). "Wedging-to-E" is a mapping of the type f: $S \rightarrow S$ where S is the 12 pitch-classes. The wedging-to-E transformation is not an operation because in the right column of Figure 2c some ele-

(a)	(b)	(c)	(d)	(e)
$f: S \to S'$	**T_3**	**“wedging-to-E”**	**Q_3**	**$I_4 * T_3$**
C → pp	C → E♭	C → C♯	C → D♯	C → E♭ → C♯
C♯ → p	C♯ → E	C♯ → D	C♯ → A♯	C♯ → E → C
D → mp	D → F	D → E♭	D♯ → F♯	D → F → B
E♭ → ff	E♭ → F♯	E♭ → E	E → C♯	E♭ → F♯ → B♭
E → ff	E → G	E → E	F♯ → A	E → G → A
F → ff	F → A♭	F → E	G → D♯	F → A♭ → A♭
	F♯ → A	F♯ → F	A → C	F♯ → A → G
	G → B♭	G → F♯	A♯ → G	G → B♭ → F♯
	A♭ → B	G♯ → G		A♭ → B → F
	A → C	A → G♯		A → C → E
	B♭ → C♯	B♭ → B♭		B♭ → C♯ → E♭
	B → D	B → C		B → D → D

Figure 2. Mapping tables

ments of S appear more than once and some elements appear not at all. However it is a mapping from a set to itself, so it is correctly termed “transformation.”

Case 2, f: (S × S)→S. Binary composition. This mapping type involves *Cartesian products*. *Let S and S′ be sets. The Cartesian product S × S′ is the set of all ordered pairs (s, s′) such that s is a member of S and s′ is a member of S′* (GMIT 1). With this definition we can write the Cartesian product of any two sets. Let S = {C♯, D} and S′ = {E♭, E, F}. The Cartesian product is the set of all pairs as follows: (S × S′) = {(C♯, E♭), (C♯, E), (C♯, F), (D, E♭), (D, E), (D, F)}. In these pairs, the order of elements matters—hence the term “ordered pairs” in the definition. For example, (C♯, D) is not the same element as (D, C♯). Armed with the notion of a Cartesian product, we can now consider expressions such as (C♯, D)→ E. In this expression, the arrow denotes *binary composition*, a term whose meaning we consider next.

So far we have informally treated the word “group” as if it denotes only a set. Actually “group” denotes a two-part entity composed of (1) a set of group elements and (2) a “law of composition” that tells us what happens when we multiply one group element with another. Lewin refers to this law as a *binary composition*, or *BIN*. A *group* is formally defined as *a set G together with a law of composition which is associative and has an identity element, and such that every element of G has an inverse* (Artin 42). What is meant by “identity,” “inverse,” and “associative” will be considered later. For now, we focus on “binary composition.”

Specifically, a binary composition is a law that tells us what the

	$\mathbf{T_0}$	$\mathbf{T_4}$	$\mathbf{T_8}$
$\mathbf{T_0}$	T_0	T_4	T_8
$\mathbf{T_4}$	T_4	T_8	T_0
$\mathbf{T_8}$	T_8	T_0	T_4

Figure 3a. A group table

	X_0	X_4	X_8
X_0	X_4	X_0	X_8
X_4	X_8	X_4	X_0
X_8	X_0	X_8	X_4

Figure 3b. Not a group table

answer "c" will be for any expression of the form (a * b). In these expressions the star (*) denotes binary composition. Atonal theory, for example, tells us that the result of combining T_1 with T_1 is T_2, which is to say $(T_1 * T_1) \rightarrow T_2$. Binary composition can describe the course of events in a piece of music.[11] A chord might be (a) transposed, then (b) inverted, and then (c) transposed again as a piece unfolds. The final state of the chord would be "a transposed inversion of a transposition," the result of "composing"—in the sense of "putting together"—(a) with (b) with (c). When the context is clear the symbol denoting binary composition (such as *, x, or •) is sometimes omitted and replaced with parentheses or with nothing: a * b = (a)(b) = ab.[12]

The binary composition for small groups is usually displayed (and thereby defined) via a *group table*. A group's binary composition is completely determined by its group table. For any pair of group elements the table will show the result of their combined action. For example, from Figure 3 we can conclude that $T_4 * T_8 = T_0$ and that $T_8 * T_8 = T_4$.

The binary composition of group operations is an instance of the mapping type f: $(S \times S) \rightarrow S$. The second S is not S′ (as in the definition of a Cartesian product) because only one set, viz. S, is being referenced. Precisely, *a binary composition on X is a function BIN that maps X × X into X. We write BIN (x, y) for the value of BIN on the pair (x, y).* (GMIT 5.) Using this definition, in Figure 4 we rewrite the group table in Figure 3. G = $\{T_0, T_4, T_8\}$, and its Cartesian product—the set of all its pairs—has

$$
\begin{array}{lcl}
(T_0, T_0) & \rightarrow & T_0 \\
(T_0, T_4) & \rightarrow & T_4 \\
(T_0, T_8) & \rightarrow & T_8 \\
(T_4, T_4) & \rightarrow & T_8 \\
(T_4, T_8) & \rightarrow & T_0 \\
(T_8, T_8) & \rightarrow & T_4 \\
(T_8, T_4) & \rightarrow & T_0 \\
(T_4, T_0) & \rightarrow & T_4 \\
(T_8, T_0) & \rightarrow & T_8
\end{array}
$$

Figure 4. Mapping of the type BIN: $(G \times G) \rightarrow G$

nine elements: $\{(T_0, T_0), (T_0, T_4), (T_0, T_8), (T_4, T_4), (T_4, T_8), (T_8, T_8), (T_4, T_0), (T_8, T_0), (T_8, T_4)\}$. A group table supplies a convenient way to display how BIN maps every Cartesian pair from $(G \times G)$ onto a single element in G.

It is a mistake to assume that group elements such as T_4 are individual numerical values and that combining any pair of group elements amounts to applying ordinary addition, subtraction, or multiplication to values in accordance with an equation. This misconception creates confusion in the study of transformations other than the usual pitch-class transpositions and inversions. It should be kept in mind that *each element in a group table is not an isolated value but rather corresponds to a mapping table.*[13]

Figure 2d illustrates this assertion in an unfamiliar context. It shows the mapping table for a group element Q_3 which acts on the octatonic space S = {C, C♯, D♯, E, F♯, G, A, A♯}. (Q_3 is a member of a group named "$STRANS_2$" which we will examine later in this article.) The action of Q3 cannot be summarized via a simple rule of adding or subtracting pc-numbers. For example, though Q_3 sends C "up" three semitones to D♯, it sends G "down" three semitones to E. To understand Q_3 one needs the entire table and keeping a single value in mind will not suffice.

The multiplying of group elements, then, might be regarded not as a combining of individual values but rather as a combining of mapping tables. A procedure of combining tables is illustrated in Figure 2e, which defines the binary composition of T_3-transposition followed by I_4-inversion. The action of doing T_3 first and I_4 second is written algebraically as "$I_4 * T_3$" because according to the convention of *left orthography* one performs T/I group operations from right to left. This convention is visible in the following styles of notation, all of which may be encountered in the literature. In each, T_3 is applied to D first, after which I_4 is applied.

$I_4 * T_3$: D→ B.

$(I_4 * T_3)(D) = B$.

$I_4(T_3(D)) = B$.

In music theory this convention is not absolute. Neo-Riemannian operations furnish a notable exception since they are often performed from left to right (Cohn 1997, 23). However, since we will later need to combine T/I with neo-Riemannian operations, we will use left orthography (right to left performance) for all operations in this paper. In each row of Figure 2e the first arrow shows the action of T_3 and the second arrow shows the action of I_4. The table shows that $(I_4 * T_3)$ applied to, say, D results in B. By combining mapping tables in this fashion we can interpret the action of composite operations.

A binary composition is associative if for group elements a, b, and c, (a * b) * c = a * (b * c). Roughly speaking, if you can move parentheses without changing the result then a binary composition is associative. Thinking of binary composition as a combining of operation tables makes it easy to see *group multiplication is associative* (as stipulated in the definition of a group). In a row of a combined table whether we chase a sequence of three arrows as (→→)→ or→ (→→), the arrows' destination will be the same.[14]

Case 3. f: (S × S)→IVLS. The interval function of a GIS. GMIT describes the interval function as int: (S × S)→IVLS, where S is the space of a GIS and IVLS is the set of intervals of a GIS. An interval function may be defined creatively to fit the particulars of a given musical context.

Sentences (7)–(10) illustrate the idea of an interval function in a familiar context.

(7) int: (C♯, D)→ minor second.

(8) (E♭, G)→ major third.

(9) (C, F♯)→ augmented fourth.

(10) (C, C)→ unison.

Sentence (7) reads "the interval function sends the pair (C♯, D) to the interval of a minor second."

One must be careful not to confuse (S × S)→IVLS with (S × S)→S. The former involves two distinct sets (S and IVLS) whereas the latter involves only the one set S. Habits from arithmetic can lead to the mistaken assumption that whatever appears on the right-hand side of an equation is the same kind of element as whatever appears on the left (as is the case in expressions such as 2 + 4 = 6). Sentences (11)–(14) are examples from existing theory in which (S × S)→IVLS might be confused with (S × S)→S.[15] In each, the map is (S × S)→S′, despite the resemblance between the entities on both sides of the equal sign. Though it appears that the "4" on both sides of the equation in (11) is the same kind of entity, the "4" on the right-hand side is from IVLS whereas the "4" on the left hand side is from S. Similarly, in (12) at first glance it might appear that (8, 8) is the same kind of entity as (1, 4), and in (13) that (1011) is the same kind of entity as (0100). Aware of the potential for confusion, Lewin notates (13) as (14) (introducing the symbols "@" and "< >") to prevent the confusion of S with S′.

(11) int(0, 4) = 4. (set of pitch-classes→ group of intervals).

(12) int((1, 4), (9, 12)) = (8, 8). (Durations→ intervals).

(13) int((0100), (1111)) = (1011). (musical states to musical transformations).

(14) int((@0100), (@1111)) = <1011>.

Sentences (11)–(14) also illustrate that an interval function is not limited to the familiar intervallic context of (7)–(10) but may be applied to any kind of musical space.

III. Musical Intuitions and Group Structure.

For a set of operations to be a group, *associativity* (already discussed), *identity*, *closure*, and *inverse* conditions must be satisfied (GMIT 3–6). In music analysis, the inverse condition insures that if you move from one point to another in a musical space you will have a way to get back to the original point. Since music involves moving towards goals as well as moving away from starting points, it is intuitively desirable to have a system that allows you to treat any musical object as either a beginning or an ending. For example, if one moves from C to G, one would like to have a way to return to C.

To find the inverse of a particular operation, reverse the direction of arrows in the operation's mapping table. The table for T_4 in Figure 5a is followed in 5b by the table for T_4's inverse. Notice that this inverse sends E to C, F to C♯, and so forth. As can be confirmed by comparing Figures 5b and c, the inverse of T_4 is the operation T_8. The *inverse condition* requires that the inverse of each group element must be in the group, so any group containing T_4 must also contain T_8, T_4's inverse.

If you multiply any operation by its inverse, the result is you "go nowhere" (Figure 5d). For example, the net result of (T_8 * T_4) is that C gets sent to C, C♯ to C♯, and so forth. The operation that fixes each point without sending anything anywhere is termed the *identity* and is desig-

(a)	(b)	(c)	(d)	(e)
T_4	**T_4's inverse**	**T_8**	**T_8 * T_4**	**T_0**
C → E	C ← E	E → C	C → E → C	C → C
C♯ → F	C♯ ← F	F → C♯	C♯ → F → C♯	C♯ → C♯
D → F♯	D ← F♯	F♯ → D	D → F♯ → D	D → D
E♭ → G	E♭ ← G	G → E♭	E♭ → G → E♭	E♭ → E♭
E → A♭	E ← A♭	A♭ → E	E → A♭ → E	E → E
F → A	F ← A	A → F	F → A → F	F → F
F♯ → B♭	F♯ ← B♭	B♭ → F♯	F♯ → B♭ → F♯	F♯ → F♯
G → B	G ← B	B → G	G → B → G	G → G
A♭ → C	A♭ ← C	C → A♭	A♭ → C → A♭	A♭ → A♭
A → C♯	A ← C♯	C♯ → A	A → C♯ → A	A → A
B♭ → D	B♭ ← D	D → B♭	B♭ → D → B♭	B♭ → B♭
B → E♭	B ← E♭	E♭ → B	B → E♭ → B	B → B

Figure 5. Inverse and identity operations

nated by convention as "*e*." The *identity condition* requires that a group contain the identity element. In the T/I group, the identity element is usually labeled T_0 (Figure 5e).

The *closure condition* requires the result of multiplying any two group elements to be an element of the group. This condition is not automatically fulfilled by any set of operations. For example, consider the set of operations $X = \{T_0, T_3, T_9, T_1, T_{11}\}$. Here the identity element is included, T_0. The inverse of every element is also included: the inverse of T_0 is T_0, the inverse of T_3 is T_9 (and vice versa), and the inverse of T_1 is T_{11} (and vice versa). So X satisfies two conditions of a group structure. However, consider what happens when we multiply T_1 by T_3. The result is T_4, an operation that is not included in the set X. The product of two elements has produced an element that is *not* in the parent set, viz. X, so the set of operations does not satisfy the closure condition and is therefore not a *group* of operations.

What musical intuition is engaged by the closure condition? Once we recognize that a piece uses a particular musical motion, it is intuitively attractive to consider all pathways that motion makes available. For example, the motion of "modulating by major third upwards" includes the possibility of moving not only from C major to E major, but via double motion from C major to G♯ major. A composer may not use all available pathways but rather may select only a few. On the other hand, a composer may in the course of a work systematically realize all the possibilities made available by a move seen early on. Understanding the possibilities of a given motion enhances music appreciation: if one knows all the roads that are not taken, one can better appreciate the roads that are.

To see if a set of interesting operations forms a group one needs to verify that all group conditions are satisfied. Cayley's observation gives us a trick that can be used to exclude non-groups from consideration if the number of operations is not large. Cayley observed *a table for a group will display each element of the group exactly once in each column and each row*. An occurrence of the identity in the body of a group table will correspond to the multiplication of an element with its inverse, and an occurrence of a caption of a row or column within the body of the same row or column corresponds to a multiplication by the identity. The absence of foreign entries demonstrate closure. Figure 3a supplies an illustration. The occurrence of T_0 in the table's second row, third column corresponds to the multiplication of T_8 with its inverse, T_4; the occurrence of the caption of row three—namely T_8—in the table's first column, third row, indicates multiplication by the identity (T_0). Although Cayley's observation gives us a shortcut means for identifying non-groups, a disclaimer is in order: while any group table necessarily satisfies Cayley's observation, a table that satisfies Cayley's observation is not necessarily a group table. In Figure 3b, for example, while each element appears exactly once in each column and once in each row, there is no identity element so {X_0, X_4, X_8} is *not* a group.

The next section uses the preceding information on group tables and interval functions to illustrate generalized interval systems and STRANS systems, which correspond to Lewin's Cartesian and transformational perspectives respectively.

IV. A Cartesian Perspective.

A central point of GMIT is that the GIS and STRANS perspectives (despite differing philosophical emphases) are at bottom equivalent formulations. Formally, this means that any STRANS system will have a particular GIS that matches it and vice versa. It also means if a system is not simply transitive—say, the T/I group of atonal theory acting on pitch classes—it cannot support a GIS.

To demonstrate the underlying sameness of the GIS and STRANS perspectives, this section will derive a GIS system, GIS_1, from a particular STRANS system, $STRANS_1$. ("STRANS" and "GIS" are general terms, "$STRANS_1$" and "GIS_1" are specific terms.) This derivation will make tangible the definition of a GIS.

Figures 6 and 7 show the mapping and group tables for the $STRANS_1$ system (GMIT 251). Recall that an STRANS system requires an S and a G. Here S = {C, C♯, D♯, E, F♯, G, A, A♯}, and G = {R_0, R_3, R_6, R_9, K, L, M, N}. An STRANS system also requires fulfillment of the simply transitivity condition (Definition 1). It can be verified that this condition if fulfilled by $STRANS_1$; for instance, R_3 is the unique operation that sends D♯ to F♯, and K is the unique operation that sends E to A.[16]

Using Definition 2 as a guide, our agenda now is to translate the $STRANS_1$ system into the GIS_1 that matches it. In order to accomplish this translation, we need to spend some time with the definition of a GIS (Definition 2 below). The first part of the definition combines items considered earlier: a musical space, a mathematical group, and an interval function. Added to these in the second part of the definition are two conditions.

Definition 2. A Generalized Interval System (GIS) is an ordered triple (S, IVLS, int), where S, the space of the GIS is a family of elements, IVLS, the group of intervals for the GIS, is a mathematical group, and int is a function mapping S × S into IVLS, all subject to the two conditions (A) and (B) following:

*(A): For all r, s, and t in S, int(r, s) * int(s, t) = int(r, t).*

(B): For every s in S and every i in IVLS, there is a unique t in S which lies the interval i from s, that is a unique t which satisfies the equation int(s, t) = i. (GMIT 26).

R_O	R_3	R_6	R_9	K	L	M	N
C → C	C → D♯	C → F♯	C → A	C → C♯	C → E	C → G	C → A♯
C♯ → C♯	C♯ → E	C♯ → G	C♯ → A♯	C♯ → C	C♯ → D♯	C♯ → F♯	C♯ → A
D♯ → D♯	D♯ → F♯	D♯ → A	D♯ → C	D♯ → A♯	D♯ → C♯	D♯ → E	D♯ → G
E → E	E → G	E → A♯	E → C♯	E → A	E → C	E → D♯	E → F♯
F♯ → F♯	F♯ → A	F♯ → C	F♯ → D♯	F♯ → G	F♯ → A♯	F♯ → C♯	F♯ → E
G → G	G → A♯	G → C♯	G → E	G → F♯	G → A	G → C	G → D♯
A → A	A → C	A → D♯	A → F♯	A → E	A → G	A → A♯	A → C♯
A♯ → A♯	A♯ → C♯	A♯ → E	A♯ → G	A♯ → D♯	A♯ → F♯	A♯ → A	A♯ → C

Figure 6. Mapping tables for the $STRANS_1$ operations.

	R_0	R_9	R_6	R_3	K	L	M	N
R_0	R_0	R_9	R_6	R_3	K	L	M	N
R_3	R_3	R_0	R_9	R_6	N	K	L	M
R_6	R_6	R_3	R_0	R_9	M	N	K	L
R_9	R_9	R_6	R_3	R_0	L	M	N	K
K	K	N	M	L	R_0	R_3	R_6	R_9
L	L	K	N	M	R_9	R_0	R_3	R_6
M	M	L	K	N	R_6	R_9	R_0	R_3
N	N	M	L	K	R_3	R_6	R_9	R_0

Figure 7. Group table for $STRANS_1$ (= SIMP)

Comparing the requirements of each system reveals similarities. Both systems require a group and a space as specified in the first two rows of Figure 8. But a GIS involves an additional entity, the interval function, int, and a Condition A which asks that intervals combine in a particular way. The interval function is the third entity "int" in (S, IVLS, int), the ordered triple of Definition 2.

Condition A conforms to our intuitions about combining intervals, as shown in Figure 9. It states essentially this: Given the configuration in

STRANS	GIS
1. A group STRANS.	1. A group IVLS.
2. A space S.	2. A space S.
3. The action of the group on the space must be simply transitive.	3. For every s in S and every i in IVLS, there is a unique t which lies the interval i from s. (GMIT 26, Condition B).
	4. An interval function (designated "int") which maps each ordered pair of objects from the space into an element of the group IVLS.
	5. Group multiplication in IVLS satisfies the equation int(s, t) * int(t ,u) = int(s, u). GMIT 26, Condition A.)

Figure 8. Requirements of STRANS and GIS systems

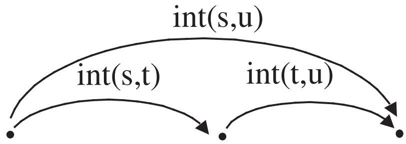

Figure 9. Condition A of a GIS

Figure 9, if we compose the interval s-to-t with the interval t-to-u, the resultant compound interval will be s-to-u, the interval of the large arc that results from composing the two smaller arcs. The intuition captured by this spatial metaphor is so obvious as perhaps to make the metaphor dispensable when thinking of usual situations such as combining a minor third with a perfect fifth. However, visualizing s, t, and u in this way becomes something useful to grab onto when working with the general idea of interval as a move of any kind in a musical space of any kind.

Algebraically, Condition A amounts to a definition of the binary composition of the group of intervals, IVLS. Recall that we defined a group as a set together with a binary composition. The common dictionary meaning of "compose" is "to form by putting together." This sense is engaged by Condition A. The question "what is the binary composition of the group IVLS" can be restated as "how does one put together the intervals of IVLS?" The composition of intervals is just what "int(s, t) * int(t, u) = int(s, u)" describes.

We now begin deriving our GIS_1 from $STRANS_1$ by interrelating intervals in GIS_1 with operations in $STRANS_1$. GIS_1 will have eight intervals, which we label in Figure 10a with letters g, h, i, j, k, l, m, and n. (Why only eight intervals are needed will be explained later.) The table shown for GIS_1's interval function is not complete; a complete table would have an entry for each of the 64 possible pairs from S. Though incomplete, this table will suffice for the purposes at hand.

Next consider the second entry of Figure 10a: int(C, D♯) = h. Using our technique of noun-to-verb paraphrase, we can restate (15) as (16):

(15) "The distance from C to D♯ is the interval h."

(16) "if we h-transpose C we arrive at D♯."

Paraphrasing in this way gives us the entry of the second column, second row, in which the second pitch is described as a transposition by h. The rest of the second column is similarly derived.[17] The basic idea behind translating an STRANS into a GIS system is captured by the substitution in Figure 10c: *The elements of an STRANS group are transpositions by intervals of the corresponding GIS*. In $STRANS_1$, then, transposition by the GIS_1 interval g is the operation R_0, transposition by the GIS_1 interval h is R_3, and so forth. The legend in Figure 10d summarizes the equivalences between $STRANS_1$ operations and transpositions in GIS_1. In retrospect, we can see why there are only eight intervals in the GIS we are seeking. In any GIS the number of intervals will be the same as the number of ways to transpose.

We seem to have the GIS_1 we want. We have the three components the definition of a GIS requires: a space S, a group of intervals IVLS = {g, h, i, j, k, l, m, n}, and an interval function int. But before we can conclude GIS_1 is a generalized interval system, we need to verify that it satisfies conditions A and B of Figure 8.

Does the interval function of GIS_1 fulfill condition A, the requirement that intervals combine in an intuitively natural fashion as diagrammed in Figure 9? Let us explore this question by testing Condition A with the example shown in (17). Condition A tells us that (17) should be true. Using the legend from Figure 10 we can incorporate the letters for intervals. Sentence (20) rewrites (17) using the interval names in (18) and (19).

(17) int(D♯, A) * int(A, E) = int(D♯, E) = m.

(18) int(D♯, A) = i

(19) int(A, E) = k.

(20) i * k = m.

(21) Tk(Ti(s)) = Tm(s).

(22) (K * R6)(s) = M(s).

To see if condition A is satisfied, we must show that (20), a rewriting of (17), is actually true. (20) says that if you k-transpose the i-transpose of

(a)	(b)	(c)	(d)
GIS_1's interval function	**Renaming pitches as transpositions**	**Renaming transpositions as operations from $STRANS_1$**	**Legend**
int (C, C) = g	= int (C, T_g(C))	= int (C, R_0(C))	$T_g = R_0$
int (C, D♯) = h	= int (C, T_h(C))	= int (C, R_3(C))	$T_h = R_3$
int (C, F♯) = i	= int (C, T_i(C))	= int (C, R_6(C))	$T_i = R_6$
int (C, A) = j	= int (C, T_j(C))	= int (C, R_9(C))	$T_j = R_9$
int (C, C♯) = k	= int (C, T_k(C))	= int (C, K(C))	$T_k = K$
int (C, E) = l	= int (C, T_l (C))	= int (C, L(C))	$T_l = L$
int (C, G) = m	= int (C, T_m(C))	= int (C, M(C))	$T_m = M$
int (C, A♯) = n	= int (C, T_n(C))	= int (C, N(C))	$T_n = N$

Figure 10. GIS intervals and STRANS operations

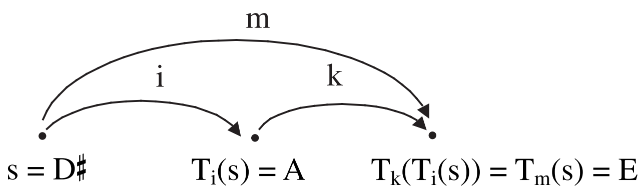

Figure 11

any note in S, say D♯, you should get the m-transpose (Figure 11). In other words, it says (21).

But since (per Figure 10d) we know that T_i = R6, and T_k = K, sentence (21) can be rewritten as (22). And we know that (22) is true from the group table of $STRANS_1$ (Figure 7). Since we know (22) is true, we know that (17) is true. So in (17)–(22) we have shown condition A to be satisfied. Using this kind of reasoning, we can verify in general that condition A is satisfied by GIS_1.

Is Condition B satisfied by GIS_1? Condition B asks that for every s in S and every i in IVLS, there is a unique t in S which lies the interval i from s—that is, a unique t which satisfies the equation int(s, t) = i. To test Condition B, we need to start with an s in S and an i in IVLS, so we select D♯ from S and k from IVLS. We can write the question Condition B asks of the test case as (23).

(23) "what t is the interval k from D♯, and is this t unique?" (int(D♯, t) = k).

Taking advantage of our technique of noun-verb paraphrase (p. 102), we rewrite (23) as (24):

(24) "if we k-transpose D♯, what t results, and is this t unique?" (T_k(D♯) = t).

As the definition of GIS transposition implies, for any s, t in S and any k in IVLS, *int(s, t) = k if and only if $T_k(s) = t$.* So we can answer the question in (23) by considering (24). Because $STRANS_1$ is simply transitive, T_k is the only transposition that sends D♯ to A♯ (Figure 6). Also T_k does not send D♯ to any note other than A♯ by the uniqueness requirement of a mapping. So in this case A♯ is the unique t we seek and Condition B of a GIS is satisfied. Through this reasoning it can be shown that $STRANS_1$ in general satisfies Condition B.

We have translated $STRANS_1$ into GIS_1 by defining an int function for GIS_1 and by using the relationships in $STRANS_1$'s mapping and group tables to show that conditions A and B of a GIS are satisfied. In the sense that we can translate $STRANS_1$ into GIS_1, $STRANS_1$ and GIS_1 are equivalent systems. However, they have differing conceptual emphases.

The conceptual relationship between $STRANS_1$ and GIS_1 can be explored through noun-to-verb paraphrase.

(25) (g, s)→ t.

(26) (s, t)→ i.

(27) (s, t)→ g.

Sentence (25) models the situation of a dynamic performer in a particular musical state s who chooses action g to arrive at t. In other words, the outcome of the pair (g, s) will be the transformed musical state, t. In contrast, (26) models the situation of a static observer. Before the observer's eyes are two points in musical space, s and t. The observer evaluates the pair (s, t) and notes the distance between them as i.

The difference between the subject-oriented and observer-oriented accounts dissolves if one invokes Lewin's generalized conception of interval. This vanishing of difference can be illustrated by rewriting (26) as (27). In (27) the letter g substitutes for i. Our generalized conception of interval permits us to make this substitution, because we understand that the distance-word "interval" (denoted by "i") in the GIS formulation can *also* stand for the transformation-word "group element" (denoted by "g") in the STRANS formulation. With this substitution, we see directly that the GIS and the simply transitive formulations both associate a given s and g with a particular element t.

> We tend to imagine ourselves in the position of *observers* when we theorize about musical space; the space is 'out there', away from our dancing bodies or singing voices. 'The interval from s to t' is thereby conceived as modeling a relation of *extension*, observed in that space external to ourselves; we 'see' it out there just as we see distances between holes in a flute, or points along a stretched string. . . .
>
> In contrast, the transformational attitude is much less Cartesian. Given locations s and t in our space, this attitude does not ask for some observed measure of extension between reified 'points'; rather it asks: 'If I am *at* s and wish to get to t, what characteristic gesture (e.g. member of STRANS) should I perform in order to arrive there?' The question generalizes in several important respects: 'If I want to change Gestalt 1 into Gestalt 2 (as regards to content, or location, or anything else), what sorts of admissible transformations in my space (members of STRANS or otherwise) will do the best job?' Perhaps none will work completely, but 'if only . . . ,' etc. This attitude is by and large the attitude of someone inside the music, as idealized dancer and/or singer. No external observer (analyst/listener) is needed. (GMIT 158–59.)

V. Atonal and Neo-Riemannian Systems.

The close study of the definition of a GIS gives a precise understanding of the transformational/Cartesian dichotomy central to Lewin's thought. In this section we will reap further dividends from this study as we evaluate the relationship between STRANS systems and two other kinds of system commonly encountered in the literature, neo-Riemannian systems that use parallel, leading-tone-exchange, and relative operations, and atonal systems that use pitch-class transpositions and inversions.[18]

The neo-Riemannian L/R group, the atonal T/I group, and the group of symmetries of the dodecagon all have the same structure.[19] Before demonstrating this claim we will briefly review each group. The neo-Riemannian L/R group is also called the PLR group because neo-Riemannian theory studies the three operations: P (Parallel), L (Leading-tone Exchange), and R (Relative). P sends a triad to its parallel, for instance, C major to C minor, or C minor to C major. Notice that C minor and C major both contain a common perfect fifth, C-G. Accordingly, we may say that P connects the two consonant triads that have a common perfect fifth. Similarly, L sends to the triad with a common minor third (e.g., C major to E minor), and R sends to the triad with a common major third (e.g., C major to A minor). The mapping tables for L and R are shown in Figure 12.

When it comes to finding *generating elements*[20] of the group, P can be set aside since all elements of the group can be expressed in terms of L and R. Specifically, P = R*L*R*L*R*L*R. By taking RL = R * L, we can rewrite the preceding equation as

P = RL * RL * RL * R.

Under left orthography, with the composite operation RL we perform L first and R second. Thinking in terms of RL will come in handy later as we formulate a brief definition of the neo-Riemannian L/R group. We can check this equation by applying (RL * RL * RL * R) to a specific major chord and see if P, the parallel triad, is the result.

RL * RL * RL * R (C+) = RL * RL * RL (A-) = RL * RL (D-) = RL (G-) = (C-).

The parallel minor chord, (C,-) is the result, and it can be verified in general that P = (RL * RL * RL * R). Because P can be set aside, the neo-Riemannian PLR group is perhaps better named the L/R group in direct analogy to the T/I group which also can be generated from just two elements (e.g., T_5 and I).[21]

The group of symmetries of the dodecagon include the transforma-

(a)		(b)		(c)		(d)	
L		**R**		**RL**		**I_{11}**	
C	→ e	C	→ a	C	→ G	C	→ e
c	→ A♭	c	→ E♭	c	→ f	c	→ E
D♭	→ f	D♭	→ b♭	D♭	→ A♭	D♭	→ e♭
c♯	→ A	c♯	→ E	c♯	→ f♯	c♯	→ E♭
D	→ f♯	D	→ b	D	→ A	D	→ d
d	→ B♭	d	→ F	d	→ g	d	→ D
E♭	→ g	E♭	→ c	E♭	→ B♭	E♭	→ c♯
d♯	→ B	d♯	→ F♯	d♯	→ g♯	d♯	→ C♯
E	→ g♯	E	→ c♯	E	→ B	E	→ c
e	→ C	e	→ G	e	→ a	e	→ C
F	→ a	F	→ d	F	→ C	F	→ b
f	→ D♭	f	→ A♭	f	→ b♭	f	→ B
F♯	→ a♯	F♯	→ d♯	F♯	→ C♯	F♯	→ b♭
f♯	→ D	f♯	→ A	f♯	→ b	f♯	→ B♭
G	→ b	G	→ e	G	→ D	G	→ a
g	→ E♭	g	→ B♭	g	→ c	g	→ A
A♭	→ c	A♭	→ f	A♭	→ E♭	A♭	→ a♭
g♯	→ E	g♯	→ B	g♯	→ c♯	g♯	→ G♯
A	→ c♯	A	→ f♯	A	→ E	A	→ g
a	→ F	a	→ C	a	→ d	a	→ G
B♭	→ d	B♭	→ g	B♭	→ F	B♭	→ f♯
a♯	→ F♯	a♯	→ C♯	a♯	→ d♯	a♯	→ F♯
B	→ e♭	B	→ g♯	B	→ F♯	B	→ f
b	→ G	b	→ D	b	→ e	b	→ F

Figure 12. Comparison of T/I with L/R permutations

tions which rotate or “flip over” the dodecagon in such a manner that its position in space remains fixed. It is commonly referred to as “D12,” where D stands for “dihedral.”[22] Generating operations are σ (a rotation by 1/12 of a turn, by “one hour” on a clockface), and τ (a flipping over of the dodecagon in place such that its back becomes its front).

Since we will be deriving the L/R and dodecagon group from the T/I group, let us first check that T/I is indeed a group. T/I satisfies the closure condition because the product of transpositions and inversions will always be some transposition or an inversion. It satisfies the inverse condition because every element’s inverse is in the group. For instance, the inverse of I_2 is I_2, and the inverse of T_{10} is T_2. It satisfies the identity condition because T_0 is the identity element and T_0 is in the group.

Two finite groups have the same structure—are isomorphic—if under some relabeling rule their group tables are identical.[23] Under the following relabeling procedure, the group table for T/I will become the group table for the symmetries of the dodecagon. Starting with a T/I group table, relabel T_1 as σ and I_0 as τ, Continue by relabeling elements other than T_1 and I_0 in the T/I table as compounds of σ and τ. For example, $I_2 = (T_1 * T_1 * I_0)$ would be relabeled as $(\sigma * \sigma * \tau) = \sigma^2\tau$. To arrive at the neo-Riemannian L/R group from the dodecagon group, relabel σ as RL and τ as L.

In this way we can use group tables to prove the T/I, L/R, and dodecagon groups have a common structure. Since these group tables are enormous (576 entries in each), we may not want to write the tables in their entirety but would prefer instead to imagine how they can be made equivalent. A shortcut around writing the full tables is to see if the *defining relations* of the involved groups match under the relabeling.[24] To define a group in briefest terms, one chooses a small number of elements from the group and specifies how they combine to produce the identity element. If this is done strategically, all relationships in the group can be inferred from these few items of information. Briefly defining in this way is known as giving the *defining relations* of a group. The T/I group has three defining relations on its generators, T_1 and I_0, as shown in the first column of Figure 13. As shown in the second and third columns, the defining relations are preserved in D_{12} and L/R under our relabeling of T_1 as RL as σ, and I_0 as L as τ.

Figure 13 is somewhat abstract. To directly illustrate the procedure of relabeling we can work with small subgroups.[25] The top of Figure 14 shows the T/I subgroup $\{T_0, T_6, I, T_6I\}$. Below it are the corresponding tables for subgroups of L/R and D_{12}. Comparing captions shows the relabeling used. Since their tables can be made equivalent via a relabeling rule, these subgroups are isomorphic to one another.

One might question the choice of RL as the new label for T_1 since T_1 and RL do not transform triads in the same way. T_1(C,E,G) = (C♯,E♯,G♯), whereas RL(C,+) = (G,+). T_1(C,E♭,G) = (C♯,E,G♯), whereas RL(C,–) = (F,–). We see here that the action of corresponding group elements on triads is not the same. Indeed, nothing in T/I will act like RL since RL sends major and minor triads in opposite directions (as shown in Figure 12c). However, to raise this question is to confuse the realm of verb-elements

T/I	**D_{12}**	**L/R**
$T^{12} = T_0 = e$	$\sigma^{12} = e$	$(RL)^{12} = e$
$I^2 = T_0 = e$	$\tau^2 = e$	$L^2 = e$
$T*I*T*I = T_0 = e$	$\sigma*\tau*\sigma*\tau = e$	$RL*L*RL*L = e$

Figure 13. Defining relations

Subgroup of T/I.

	T_0	T_6	I	T_6I
T_0	T_0	T_6	I	T_6I
T_6	T_6	T_0	T_6I	I
I	I	T_6I	T_0	T_6
T_6I	T_6I	I	T_6	T_0

Subgroup of L/R.

	e	$(RL)^6$	*L*	$(RL)^6*L$
e	e	$(RL)^6$	L	$(RL)^6*L$
$\mathbf{(RL)^6}$	$(RL)^6$	e	$(RL)^6 * L$	L
L	L	$(RL)^6*L$	e	$(RL)^6$
$\mathbf{(RL)^6*L}$	$(RL)^6*L$	L	$(RL)^6$	e

Subgroup of D_{12}.

	e	σ^6	τ	$\sigma^6\tau$
e	e	σ^6	τ	$\sigma^6\tau$
σ^6	σ^6	e	$\sigma^6\tau$	τ
τ	τ	$\sigma^6\tau$	e	σ^6
$\sigma^6\tau$	$\sigma^6\tau$	τ	σ^6	e

Figure 14. Isomorphic subgroups

(elements of G) with the realm of noun-elements (elements of S). Group isomorphisms take into account only the relation of transformations (verb-elements) to one another and are indifferent to choice of musical space (noun-elements).[26] If we focus on the binary composition of verb-elements—and put out of our mind for the time being the mapping tables of T_1 and RL—the relabeling works perfectly. For instance, in Figure 14, wherever T_6 appears in the upper table, $(RL)^6$ or σ^6 appears in a corresponding position in a lower table.

To explore the difference in the actions of the T/I and L/R groups on triads, we first need to clarify the relationship of their respective spaces. It is easy to think in error that the T/I atonal system and L/R neo-Riemannian system both involve the same space. But this is not so: the T/I space contains trichords (e.g., {C, E, G}) whereas the L/R space contains signed triads (e.g., (C, +)). This is not merely a labeling but a philosophical difference with theoretical and analytical implications. The T/I system construes triads as a composite entity constructed by combining pitch classes whereas the L/R system regards consonant triads ("Klangs" in Lewin's terminology) as fused entities (GMIT 175–76).

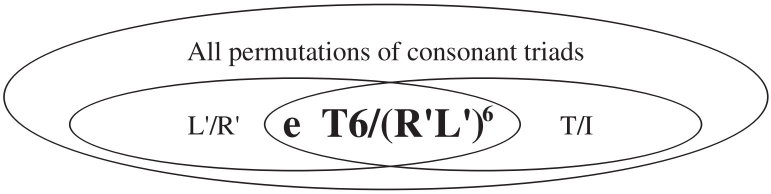

Figure 15. T/I and L′/R′ embedded in permutations
of the 24 consonant triads

To compare systems we need an L/R kind of group that acts on trichords just as T/I does. To make such a group—say L′/R′—take the space S of consonant-triad trichords such as {C, E, G}. Then invent the group G = L′/R′ with the same group table as the L/R group, except with prime marks appended to each entry (hence L/R is isomorphic to L′/R′). Write the mapping table for each operation in the L′/R′ group by rewriting the corresponding entry in L/R, showing each trichord in the form of {C, E, G} or {D, F, A}, in place of each corresponding triad in the form of (C,+) or (D,-).

We will make reference to this L′/R′ group to show how the T/I and L/R groups (though isomorphic to one another) beget distinct systems. One sometimes hears the mistaken observation that the atonal and neo-Riemannian systems are essentially equivalent—and the fact that the T/I and L/R group are isomorphic to one another may reinforce this misconception. A look at specific mapping tables shows that the T/I and L/R systems are unlike. As shown in Figure 12a, L, sends C to E minor, C minor to A♭ major, and so forth. Is there an operation from the isomorphic group, T/I, that acts like L? Since I_{11} is the unique element in T/I which matches the first row of L's action table, it is the only candidate in T/I which might match L. However, as shown in Figure 12d, I_{11} does not send C minor to A♭ major and differs from L in other rows as well. Indeed, there is no operation in T/I that permutes triads exactly as L does.

The general situation is illustrated in Figure 15. The large circle includes all the permutations of the 24 consonant triads. Each smaller circle contains the 24 operations of each group as labeled, the L′/R′ group on one hand and the T/I group on the other. Each operation within a smaller circle corresponds to a single mapping table, such as the table for, say, L′ or I_6. As shown, the only intersection between the T/I and L′/R′ circles is the identity element and T_6 (= RL^6). The point is this: *though the T/I and L′/R′ groups are isomorphic, their group actions on the 24 consonant triads engage distinct sets of permutations*. The same holds mutatis mutandis for T/I and L/R. However, despite this difference in their actions, the idea of an isomorphic relationship between them is nice to have since the T/I and L/R systems are analogous in musically significant ways.[27] Also, as will be next discussed, T/I is the *dual* of L/R and vice versa, an astonishing fact, perhaps, given the different historical origin of each system.

The foregoing illustrates that groups in transformational theory permit the study of verb-elements (G) as an independent realm, not necessarily in the context of a given set of noun-elements (S). This independence opens the door to more analytical possibilities since it allows one to apply a given group structure to different musical spaces and vice versa. *A given group may form a GIS or not, depending on how it is wedded to a space*. As the test examples in Figure 16 illustrate, the L/R (and L′/R′) and the T/I group acting on triads are simply transitive (Figure 16a-c), but the T/I group acting on pitch classes is not simply transitive and therefore cannot form a GIS or STRANS system (Figure 16d).

VI. Non-commutative Generalized Interval Systems.

The non-commutative GIS is certainly of the most interesting ideas in Lewin's work. Among other things, his work on non-commutative GIS reveals the existence of paired systems of transformations. With this work, we know that if we undertake analysis of a particular space with a GIS that is not commutative, there necessarily is a second GIS, out there somewhere, that forms a natural pair with the original GIS. The operations of this second GIS will apply to the space of the original GIS and can be combined with the operations of the original GIS to produce a larger, composite analytical system. This section works through these ideas by first sketching the theory and then offering illustrations of two non-commutative GIS systems. The first system uses a space of octatonic pitch-classes, the other a space of consonant triads. Examples from the paired system which corresponds to each of these two systems will also be given.

A GIS is non-commutative if there are two intervals i and j from its group IVLS such that $i*j \neq j*i$. In atonal theory, non-commutativity is a familiar idea since in the T/I group transpositions do not always commute with inversions. For instance $T_4*I_6 \neq I_6*T_4$. However, in GIS theory the expression $i*j \neq j*i$ takes on an unfamiliar connotation, one which we will explore in a discussion of Figure 17.

In our usual system of intervals, two intervals i and j "add up" to the same compound interval (i * j) regardless of the order in which the intervals are taken. In Figure 17a, for instance, a minor seventh results both ways. This conforms to a distance metaphor: the distance spanned by two lengths is additive and therefore not affected by the order in which the lengths are added. In contrast, order matters in a non-commutative GIS, as shown in Figure 17b. Even though we are familiar with non-commutative

(a)	*(b)*	*(c)*	*(d)*
G = T/I group.	G = L/R group.	G = L′/R′ group.	G = T/I group.
S = {the 24 consonant triads in the form of pitch-class subsets}.	S = {C+, C-, D♭+, D♭-, D+, D-, E♭+, Eb-, E+, E-, F+, F-, F♯+, F♯-, G+, G-, A♭+, A♭-, A+, A-, B♭+, B♭-, B+, B-}.	S = {the 24 consonant triads in the form of pitch-class subsets}.	S = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}.
s = {C, E, G}. t = {B, D, F♯}. I_6(s) = t.	s = C+ t = B- LRL(s) = t.	s = {C, E, G}. t = {B, D, F♯}. L′R′L′(s) = t.	s = C. t = E♭. T_3(s) = I_3(s) = t. OP(s) = t is true for *both* OP = T_3 and OP = I_3.
Is an STRANS system.	Is an STRANS system.	Is an STRANS system.	Is not an STRANS system.

Figure 16. Comparison of T/I and L/R systems

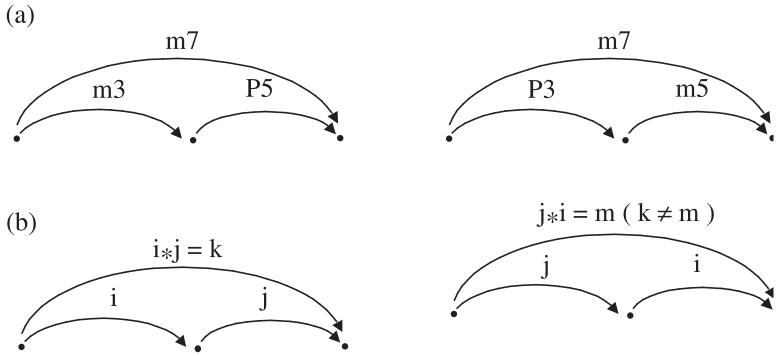

Figure 17

operations such as T_4 and I_6, the situation in Figure 17b seems unfamiliar because we normally do not think of T_4 and I_6 as *intervals* in an interval system. (Indeed, since the T/I group acting on pitch classes is not simply transitive, it is literally *not* an interval system.)

Non-commutative GISs have other features not shared by our usual system of intervals, two of which we will consider: (1) In any non-commutative GIS, not all transpositions will preserve intervals, and not all interval-preserving operations are transpositions. (2) For any non-commutative STRANS system there is a corresponding *dual* group—a second system—which is composed of just the *interval-preserving operations*. ("Dual" and "interval-preserving" will be defined shortly.)

In the familiar intervallic context we say C to G is a perfect fifth. If we transpose both C and G upwards by a minor third, we get E♭ and B♭, which also form a perfect fifth. We might say "the interval from C to G is the same as the interval from the T_3-transpose of C to the T_3-transpose of G." Paraphrasing this algebraically gives us (28). T_3 is an operation which preserves the interval of a perfect fifth: it is *interval preserving*. To make a general statement, we can rewrite (28) as (29). Sentence (29) reads "the interval from s to t is the same as the interval from the q-transpose of s to the q-transpose of t," where q is an interval in a GIS.

(28) $\text{int}(C, G) = \text{int}(T_3(C), T_3(G))$ = perfect fifth.

(29) $\text{int}(s, t) = \text{int}(Tq(s), Tq(t))$.

In a GIS if T_q (transposition-by-interval-q) satisfies the equality in (29) for any s and t in our musical space and some q in IVLS, then we say that Tq is an *interval-preserving* operation. As our C-G to E♭-B♭ example above suggests, the twelve usual transpositions are interval-preserving operations.

R_0	R_3	R_6	R_9	K	L	M	N
C → C	C → D♯	C → F♯	C → A	C → C♯	C → E	C → G	C → A♯
C♯ → C♯	C♯ → E	C♯ → G	C♯ → A♯	C♯ → C	C♯ → D♯	C♯ → F♯	C♯ → A
D♯ → D♯	D♯ → F♯	D♯ → A	D♯ → C	D♯ → A♯	D♯ → C♯	D♯ → E	D♯ → G
E → E	E → G	E → A♯	E → C♯	E → A	E → C	E → D♯	E → F♯
F♯ → F♯	F♯ → A	F♯ → C	F♯ → D♯	F♯ → G	F♯ → A♯	F♯ → C♯	F♯ → E
G → G	G → A♯	G → C♯	G → E	G → F♯	G → A	G → C	G → D♯
A → A	A → C	A → D♯	A → F♯	A → E	A → G	A → A♯	A → C♯
A♯ → A♯	A♯ → C♯	A♯ → E	A♯ → G	A♯ → D♯	A♯ → F♯	A♯ → A	A♯ → C

Figure 18. Mapping tables for the $STRANS_1$ operations

Now, imagine a musical system in which a transposition does not preserve musical intervals, and further, in which there exist interval-preserving operations that are not transpositions. That would be an unusual system indeed! Unusual as it may be, this is exactly the situation we face whenever we are working with a non-commutative GIS. Sentences (30) and (31) illustrate.

(30) int(C, D♯) = int(C, T_h(C)) = h.

(31) int(T_l(C), T_l (D♯)) = int(L(C), L(D♯)) = int(E, C♯) = int(E, T_j(E)) = j.

We arrive at (30) by consulting the mapping table for $STRANS_1$ (Figure 18) keeping in mind that $T_h = R_3$, as shown in Figure 10, column d. Sentence (31) shows what results if we transpose both C and D♯ by the interval l. (We are now speaking of transposition in GIS_1, which is not our usual type of transposition.) After l-transposing both C and D♯, the result is j. Before transposition, the interval is h, but after transposition the interval is j. We conclude T_l (transposition-by-interval-l) is *not* an interval preserving operation.

In the familiar system of intervals, transpositions are the interval-preserving operations. Sentences (30) and (31) illustrate there exist systems in which transpositions are not interval-preserving operations. This raises the question: are there operations, other than transpositions, that are interval preserving for $STRANS_1$? The answer is yes, and the operation Q_3 discussed earlier turns out to be one of them (Figure 18). The operation Q_3 takes the diminished seventh C-D♯-F♯-A and arpeggiates each of its notes "upwards." C goes to D♯, D♯ goes to F♯, F♯ goes to A, and A goes to C. But Q3 takes the diminished seventh C♯-E-G-A♯ "downwards." C♯ goes to A♯, E goes to C♯, G goes to E, and A♯ goes to G. Recall that the mapping table of an *operation* will exhibit each of the space's elements once on the sending side of the arrows and once on the receiving side. This is the case with the mapping table for Q_3 (Figure 19). So, though its action is odd, Q_3 is indeed an operation on the octatonic space of $STRANS_1$, S = {C, C♯, D♯, E, F♯, G, A, A♯}.

(32) int(C, D♯) = int(C, T_h(C)) = h.

(33) int(Q_3(C), Q_3(D♯)) = int(D♯, F♯) = int(D♯, T_h(D♯)) = h.

(34) int(C, A♯) = int(C, T_n(C)) = n.

(35) int(Q_3(C), Q_3(A♯)) = int(D♯, G) = int(D♯, T_n(D♯)) = n.

Sentences (32) and (33) test if Q_3 preserves intervals by working out the example of (30) and (31) with Q_3 instead of T_l. The answer is h in both (32) and (33) so in this instance Q3 preserves intervals. Sentences (34)

Q_3	**Q_9**	**X_1**	**X_2**	**X_4**	**X_5**	**Q_3Q_3**	**e**
C → D♯	C → A	C → C♯	C → A♯	C → E	C → G	C → F♯	C → C
C♯ → A♯	C♯ → E	C♯ → C	C♯ → D♯	C♯ → A	C♯ → F♯	C♯ → G	C♯ → C♯
D♯ → F♯	D♯ → C	D♯ → E	D♯ → C♯	D♯ → G	D♯ → A♯	D♯ → A	D♯ → D♯
E → C♯	E → G	E → D♯	E → F♯	E → C	E → A	E → A♯	E → E
F♯ → A	F♯ → D♯	F♯ → G	F♯ → E	F♯ → A♯	F♯ → C♯	F♯ → C	F♯ → F♯
G → E	G → A♯	G → F♯	G → A	G → D♯	G → C	G → C♯	G → G
A → C	A → F♯	A → A♯	A → G	A → C♯	A → E	A → D♯	A → A
A♯ → G	A♯ → C♯	A♯ → A	A♯ → C	A♯ → F♯	A♯ → D♯	A♯ → E	A♯ → A♯

Figure 19. Mapping table for the $STRANS_2$ operations

and (35) try another example and again Q3 preserves intervals. It can be verified that Q_3 is an interval-preserving operation for $STRANS_1$.

$STRANS_2$ (Figure 19) consists of *all* the interval-preserving operations for $STRANS_1$. Where do these interval-preserving operations come from? They were not in $STRANS_1$. And how do we know we have all of them? To pursue these questions we need the idea of *commuting elements*. If we have two operations, a and b such that (a * b) = (b * a), we say the "operation a commutes with operation b" and "a and b are commuting elements." Operations can commute with one another even if they are contained in a non-commutative group. For example, even though the T/I group is not commutative, we can still say that "the operation T_6 commutes with the operation I_{10}," since (T_6 * I_{10}) = (I_{10} * T_6). A theorem in GMIT (50) tells us that in a given GIS:

> *any interval-preserving operation commutes with any transposition operation.*

Algebraically we can write this theorem as $P * T_i = T_i * P$, where P is some interval-preserving operation, and Ti is a transposition by interval i, i an element of IVLS.[28] Because P commutes with any operation from STRANS, Lewin refers to the group of P-operations (of interval-preserving operations) as COMM (Lewin 1995, 105). In this context, the group of transpositions (e.g., $STRANS_1$) is referred to as SIMP, and the groups COMM and SIMP are each called the *dual* of the other.[29] SIMP is the dual of COMM, and COMM is the dual of SIMP. Figure 20 shows the group table for our example of a COMM group (the dual of the SIMP group of Figure 7). Putting together theorems 3.4.2 and 3.4.7 (GMIT 46–48) tells us that the number of transpositions is the same as the number of interval-preserving operations (since Ts and Ps are in bijective correspondence). So we know COMM in Figure 20 contains all of the interval-preserving operations for $STRANS_1$: {e, Q_3, Q_9, Q_3Q_3, X_1, X_2, X_4, X_5}.[30]

Sentences (36), (37), and (38) illustrate the claim that any element of $STRANS_2$ commutes with any element of $STRANS_1$. Figures 18 and 19 are used to work out (36)–(38).

(36) Q_3(L(C)) = L(Q_3(C)) = C♯ [L and Q_3 switch positions.]

(37) X_2(M(F♯)) = M(X_2(F♯)) = D♯ [X_2 and M switch positions.]

(38) Q_9(R_3(E)) = R_3(Q_9(E)) = A♯ [Q_9 and R_3 switch positions.]

Just from looking at the mapping tables it is not obvious that the operations of $STRANS_2$ (= COMM) are going to be the interval-preserving operations for $STRANS_1$. Indeed, the operations of COMM are unfamiliar. As shown in Figure 19, X_1 exchanges the positions of any two notes that are one semitone apart, X_2 two semitones apart, X_4 four semitones apart, X_5 five semitones. And as mentioned, Q_3 sends the C-D♯-F♯-A diminished seventh in one direction and the C♯-E-G-A♯ diminished seventh in the other direction. The composition $(Q_3{*}Q_3)$ sends a pitch to its tritone-distant partner.

Unusual as they are, these operations do form a group. Figure 20 satisfies Cayley's observation (mentioned earlier) that in a group table each element will appear precisely once in each row and each column. The mapping tables in Figure 19 show that COMM is a group which acts simply transitively on S. Therefore, it forms an $STRANS_2$ system in its own right. Further, a GIS_2 can be derived from $STRANS_2$ just as we derived GIS_1 from $STRANS_1$. So the dual group COMM gives rise to $STRANS_2$ and GIS_2 systems.

Further, a larger group (but not an STRANS system) results from combining a non-commutative STRANS with its dual. *PETEY is the family of all operations on S that can be expressed as (functionally equivalent to) something of form PT, where P is some interval-preserving operation and T is some transposition* (GMIT 57).[31] By the closure condition, any operation in the larger group PETEY must be either an element from one of the groups or an element resulting from multiplying elements of one group with elements of the other. What kind of new operations exist in the larger group? We can look at a new element by multiplying one element from $STRANS_1$ with one element from $STRANS_2$. Let us select K from $STRANS_1$ and Q3 from $STRANS_2$. Figure 21 shows the mapping table for the new operation which we will call "W," $W = (Q_3{*}K)$. The left arrow shows the action of K, the right arrow Q_3.

As the mapping table shows, the operation W maps scale segments to consonant triads and other familiar chords. For instance, {C, C♯, D♯} gets sent to an E♭ major triad, {C♯, D♯, E} to a C minor triad, {D♯, E, F♯} to a C major triad, {F♯, G, A, A♯} to an F♯ minor-seventh chord, and {C♯, D♯, E, F♯} to the C major-minor triad. The operations found in the larger group are musically suggestive, with W bringing to mind octatonic pas-

	e	**Q_3**	**Q_9**	**Q_3Q_3**	**X_1**	**X_2**	**X_4**	**X_5**
e	e	Q_3	Q_9	Q_3Q_3	X_1	X_2	X_4	X_5
Q_9	Q_9	e	Q_3Q_3	Q_3	X_2	X_5	X_1	X_4
Q_3	Q_3	Q_3Q_3	e	Q_9	X_4	X_1	X_5	X_2
Q_3Q_3	Q_3Q_3	Q_9	Q_3	e	X_5	X_4	X_2	X_1
X_1	X_1	X_2	X_4	X_5	e	Q_3	Q_9	Q_3Q_3
X_2	X_2	X_5	X_1	X_4	Q_9	e	Q_3Q_3	Q_3
X_4	X_4	X_1	X_5	X_2	Q_3	Q_3Q_3	e	Q_9
X_5	X_5	X_4	X_2	X_1	Q_3Q_3	Q_9	Q_3	e

Figure 20. Group table for $STRANS_2$ (= COMM)

C → C♯ → A♯
C♯ → C → D♯
D♯ → A♯ → G
E → A → C
F♯ → G → E
G → F♯ → A
A → E → C♯
A♯ → D♯ → F♯

Figure 21

sages by Bartók and Messiaen in which scalar melody is accompanied with tertian harmony.[32]

We now take illustrations from two familiar groups: the T/I group of atonal theory and the L′/R′ group of neo-Riemannian theory. By testing for commutativity,[33] sentences (39), (40), and (41) illustrate the claim that *the neo-Riemannian L′/R′ group is the dual of T/I when both are acting on consonant triads*.

(39) $L'(I_4(G,B,D)) = I_4(L'(G,B,D)) = (B\flat,D,F)$ [L′ and I_4 switch positions.]

(40) $L'R'(T_5(C,E\flat,G)) = T_5(L'R'(C,E\flat,G)) = (C,E\flat,G)$ [L′R′ and T_5 switch positions].

(41) $R'(I_1(A,C\sharp,E)) = I_1(R'(A,C\sharp,E)) = (C,E,G)$ [R′ and I_1 switch positions.]

We noted earlier that the neo-Riemannian L′/R′ group acting on triads is the dual of the atonal T/I group acting on triads. So the union of L′/R′ (taken as COMM) and T/I (taken as SIMP) gives us another example of a larger, PETEY group. This larger group contains new operations not included in either L′/R′ or T/I, for instance, the operation $(T_1 * L')$. At first glance, it may seem that $(T_1 * L')$ is not a new operation but merely another name for I_0 from T/I or PLR from L/R, since $(T_1 * L')$, I_0, and PLR operations all send C major to F minor, as shown in the first row of Figure 21. However, $(T_1 * L')$ is indeed a new operation and not the same as either I_0 or PLR, as revealed in the second row of Figure 22. As discussed earlier, for two operations to be considered equivalent, the entire mapping table for the first must be identical to that for the second; it is not enough that they share one entry in common.

How can one find the interval-preserving operations "out there somewhere" (like Q_3 in Figure 19) that correspond to a given non-commutative system? An important result in GMIT tells us *in a noncommutative GIS there exist transpositions that do not preserve intervals, and there exist interval-preserving operations that are not transpositions* (GMIT

$T_1(L')$	I_0	PLR
C,+ → F-	C,+ → F-	C,+ → F-
C,- → A+	C,- → F+	C,- → G+

Figure 22. Comparison of excerpts from mapping tables

50). By this we know there must exist somewhere an interval-preserving operation for $STRANS_1$ not in $STRANS_1$. Let us call this operation P.

(42) P(C) = C♯ (Step 1).

Step 1: To start our search for an interval-preserving operation P, in (42) we take P(C) = C♯.[34] Some trial-and-error is involved in this kind of start since we cannot know in advance if it will lead to an operation that is in COMM (and not just again in SIMP). All we know at this point is there are many permutations of S that send C to C♯. Our immediate goal is to see if any one such permutation is interval preserving.

Step 2: Select from our system, $STRANS_1$ an operation that sends something to C. We select C because choosing what is inside the parentheses of our equation, P(C) = C♯, is the trick that gets us the answers we want. In the mapping table for $STRANS_1$ (Figure 18) are eight choices: R_3(A) = C, R_6(F♯) = C, R_9(D♯) = C, K(C♯) = C, L(E) = C, M(G) = C, and N(A♯) = C. In (43) we arbitrarily choose R_3(A).

(43) R_3(A) = C (From group action table, Figure 18).

Step 3: In (42), which says P(C) = C♯, replace C with R_3(A) since in (43) we see R_3(A) = C. The result is (44).

(44) P(R_3(A)) = C♯ (Substituting (43) in (42)).

Step 4: The theory tells us that any interval-preserving operation is going to commute with every transposition. So by the definition of a commuting element, we may reverse the positions of P and R3.

Remember that P is the interval-preserving operation we are constructing using this method, and R_3 is a transposition from $STRANS_1$. By commutativity, we may rewrite (44) as (45).

(45) P(R_3(A)) = R_3(P(A)) = C♯ (Elements of COMM commute with elements of SIMP).

Step 5: Using the mapping tables in Figure 18, in (46) solve for P(A). So far we have determined that R_3(P(A)) = C♯. Our table tells us (46), that R3(A♯) = C♯.

(46) R_3(A♯) = C♯ (From group action table, Figure 18).

(47) P(A) = A♯ (Substituting (46) in the middle term of (45)).

Sentence (47) follows from (46) by substitution and gives us the answer we seek. We now have one more value for the operation P. We can determine another value by repeating the steps of (43)–(47) using the new piece of information we have, that P(A) = A♯. Using our trick, we again search for an operation that gives us the value inside the parentheses, in this case the pitch A. From the mapping table (Figure 18) we notice that K(E) = A. Substituting K(E) for A, we can rewrite P(A) = A♯ as P(K(E)) = A♯. Using commutativity, we can rewrite this last equation as K(P(E)) = A♯. Since K(D♯) = A♯, we conclude that P(E) = D♯. These five steps are summarized in (48)–(52).

(48) K(E) = A (From group action table, Figure 18).

(49) P(K(E)) = A♯ (Substituting (48) in (47)).

(50) K(P(E)) = A♯ (Elements of COMM commute with elements of SIMP).

(51) K(D♯) = A♯ (From group action table, Figure 18).

(52) P(E) = D♯ (Substituting (51) in (50)).

Upon reaching (52), we know that P(E) = D♯, P(A) = A♯, and P(C) = C♯. If we continue repeating our five steps we will eventually construct the mapping table for the operation X_1 of $STRANS_2$ (Figure 19). X_1 is not an element of $STRANS_1$ so we have discovered the operation P in COMM that we seek. In general, by knowing the action of the transpositions and taking advantage of the knowledge that elements of COMM commute with elements of SIMP, we can start with a GIS and find its interval-preserving operations.

Figure 23 lists some of the groups involved in GIS theory. We have seen examples of groups 1–4. If one looks only at commutative GISs, the

1. The group of intervals for the GIS (viz., IVLS).
2. The group of transpositions (TNSPS); our example was $STRANS_1$.
3.* The group of interval-preserving operations (PSVS); our example was $STRANS_2$.
4.* The group generated by PSVS and TNSPS (PETEY). We gave an example of an element from this group, element W (Figure 21).
5. The group of inversions (INVS). (See GMIT 58.)
6.* The group of interval-reversing operations. (See GMIT 58.)
7. The group generated by PSVS, TNSPS, and INVS (PETINV).

Figure 23

cases marked with an asterisk do not need to be considered separately. This is because in a commutative GIS the interval-preserving operations are the transposition operations, and the inversions are the interval-reversing operations. So in the commutative case groups 3 and 4 do not need to be considered separately since both are the same as group 2, group 6 is the same as group 5, and group 7 is the union of groups 2 and 5.[35]

Before concluding, three caveats are in order. First, although this article has focused on systems containing finite numbers of musical objects and transformations, infinite systems are covered by the theory. GMIT includes a discussion which shows how to convert an infinite into a finite set of musical objects via an *equivalence relation* and an infinite into a finite set of transformations via a *quotient group* (GMIT 7–37). Second, although this article's examples used only pitch classes and triads, the theory actually can accommodate any kind of musical object one can imagine, such as relative durations, timbral distributions, dynamics, or on-off states however construed (See GMIT 60–88 and Lewin 1995).

Third, although it was expedient to employ a temporal metaphor wherein elements are described as moving from a present into a future, actually a set of elements connected by transformational arrows is an abstract, atemporal scheme. A twelve-tone work does not necessarily present rows in the order they appear in a row table and might not present all the rows of the table. A transformational network is like a row table in being a scheme which displays "out of time" the universe of possibilities engaged by a given work. In a transformational network, the direction of the arrows specify the directionality of moves, not necessarily the temporal order of moves. For instance, an arrow for the transformation T_2 would point towards D from C. To reverse the arrow would mean something other than T_2, since pointing towards C from D would be T_{10}. So an arrow for T_2 is directional—though not necessarily in a temporal sense.[36] These temporal considerations are taken up in GMIT's study of transformational networks (GMIT 209–19).

VII. Conclusion.

We conclude with a few notes on the scope of GMIT's theory.[37] The notion of interval has been central to how composers and listeners have "conceptually structured" sound—at least in the Western art music tradition—and Lewin's work increases the tradition's scope by addressing diverse musical phenomena in intervallic terms.[38] But while GIS theory makes all kinds of intervallic readings available, their is no claim that all theoretically possible readings will be insightful.[39] For instance, formalizing a particular rhythm as a GIS interval does not automatically invest insightfulness into the description. Because a compelling interpretation of musical perceptions is needed to turn a theoretically true statement into a meaningful statement, analytical judgment plays a central role in meaningfully reducing the scope of the theory.[40] So although it may appear that the mathematics of Lewin's work is a language of scientific positivism, the emphasis on perceptual context and interpretation actually distances GMIT's theory from scientific theory—at least the kind of "covering law" theory often cited in connection with scientific research.[41]

A music theory for communicating perceptions and intuitions locates music in experience and not in nature. Mathematician Reuben Hersh similarly notes how mathematics is interior yet social: "The study of the lawful, predictable parts of the physical world has a name. That name is 'physics.' Study of the lawful predictable parts of the social-conceptual world has a name. That name is 'mathematics'" (Hersh 593). While it is misleading to regard GMIT's mathematics as serving the ends of a strong scientific explanationism,[42] it is revealing to consider it from a "social-conceptual" perspective—particularly as serving the ends of communication.[43] In minimizing the kind of misunderstanding that arises from ambiguity of terms,[44] GMIT's mathematics gives us relatively stable terms for exchange in musical culture.

The terms of GMIT's theory are precise with respect to the theory but open-ended in connotation through metaphoricity. Lewin writes "When we describe the ways in which musical sound seems conceptually structured, categorically prior to any one specific piece, we nevertheless intend our conceptual sound-worlds to be rich in potential metaphors for analyzing specific pieces. At least most of us do. . . . It is not a question of our intending metaphorical discourse or not, when we bring a theory to an analysis. We cannot help it" (Lewin 1991, 118). Metaphoricity leaves open the possibilities for how designative terms of the theory may be linked to perceptual contexts and lends a wide scope to the theory's analytical application. Wide scope is also given by the possibility of multiple, equally sturdy readings of a given passage.[45] In applying GMIT's theory in analysis, one judiciously selects both a musical space and set of transformations to capture a musical intuition. A passage that stimulates more than one intuition may inspire multiple descriptions, perhaps each with a different choice of space and transformations.[46]

Generalizing the notion of interval, encouraging metaphoricity and multiple description, and emphasizing perceptual context and intuition give us a theory with many analytical possibilities, many of which have yet to be realized. I hope this article sparks new analytical work and encourages dialogue between transformational specialists and non-specialists. Of course there is more to GMIT than the few formal concepts covered herein and more points of entry to the work than the one offered here. The rationale behind this article's focus is a view of GMIT that I share with John Clough:

[I]t would be a great mistake to assume that the mathematics is in any sense superfluous, for it is the very essence of Lewin's accomplishment to have captured, formally, a means of supporting the wealth of musical intuitions that drive his analytical quests. . . . . For the theorist who wishes to try Lewinesque analysis or theory construction, the importance of understanding the foundations of his work cannot be overstated. (Clough, 1989, 227).

NOTES

1. I thank David Clampitt, Thomas Fiore, Andrew Mead, and Robert Morris for discussions that shaped the preparation of this article.
2. Lewin 1987.
3. This article is informal in that claims are not proved but rather illustrated by example. The location of proofs and formal definitions is indicated by references to GMIT or Armstrong 1988.
4. For an introduction to the T/I operations of twelve-tone theory, see Rahn 1980, 40–55. For an introduction to neo-Riemannian theory, see Cohn 1998. The neo-Riemannian L/R group studied in this article uses the PLR family of operations. A study of the PLR family may be found in Cohn 1997. Morris 1995 compares T/I with Lewinian-transformational readings.
5. In this article, I have chosen to use the term "system" for both generalized interval systems and two-component systems composed of a group together with a set on which the group acts in a simply transitive manner. See Armstrong 91 for an exposition of a simply transitive group action. For more on the theory of systems see Satyendra 2002.
6. As will be discussed soon, a group is a composite entity that combines a set and a binary composition. It serves present purposes to focus on just the set component of a group. Though I describe a musical space as a set, GMIT describes it as a "family." Lewin reserves the term "set" to denote finite subsets of a musical space. At times it is useful to have both terms "family" and "set." Compare GMIT pp. 1 and 88 to see the context in which Lewin distinguishes between the usage of these terms.
7. In this article all references to pitch classes and triad roots assume enharmonic equivalence.
8. For more on this point see GMIT p. 180, where triadic-network transformations are not considered intervals because they do not satisfy the simple transitivity condition.
9. Homomorphisms, isomorphisms, and anti-isomorphisms are also cases in which mappings may occur between unlike sets. All three arise in GIS theory. See GMIT 13–15. Also Cohn functions employ mappings between unlike sets (Lewin 1996).
10. Parentheses notation is used in Morris 1987 and Rahn 1980.
11. See the discussion on page 134 for a caveat about the temporal interpretation of transformations.
12. The symbol "+" is reserved for binary composition which is commutative.
13. Here I take advantage of Cayley's theorem which permits me to represent each group element by a permutation table. This license amounts to considering a given group not as itself, so to speak, but in terms of another group, a group of permutations in which it is may be embedded via a homomorphism. This is a mathematical subtlety that does not affect the substance of our discussion (Armstrong 41).
14. Armstrong 26 shows this claim algebraically.
15. See in connection with (11) Rahn 1980, 25, (12) GMIT 75–76, and (13) & (14), Lewin 1995, 90.
16. R_0, R_3, R_6, and R_9 of $STRANS_1$ resemble T_0, T_3, T_6, and T_9 of T/I respectively. However, though the R operations resemble the T operations, they are not equiv-

alent. That is because the R operations act on an eight-element space whereas the T operations act on a twelve-element space. This becomes clear if one writes the table for any T operation. The table for a T operation will have twelve rows whereas the table for an R operation will have only eight rows. K, L, M, and N similarly resemble I_1, I_4, I_7, and I_{10}, though similarly need to be labeled differently.

17. In a GIS, transposition by T_i is defined int(s, T_i(s)) = i, where i is an element of IVLS (GMIT 46).
18. Hook 2002 provides theory and nomenclature for studying the larger group of uniform triadic transformations of which neo-Riemannian operations are a subgroup.
19. Since atonal theory commonly uses a circle with twelve points on the circumference as an illustrative tool, we include a consideration of the symmetries of the dodecagon. Neo-Riemannian theory also uses the symmetries of the dodecagon as a geometric representation (Clough 1998).
20. If we take just the two T/I elements T_5 and I_0 and list the possibilities of their combination we end up with the full set of twelve inversions and twelve transpositions. Since T_5 and I_0 are all we need to get the full T/I group, a mathematician would say "T_5 and I_0 are *generators* for the T/I group." A group is said to be *generated* by two elements a and b if any element in the group can be written as a product of powers of a and powers of b. Elements a and b are called *generators*. Describing a group in terms of its generators is the subject of *group presentations* (Armstrong 166).
21. Clampitt 1997 introduces the "L/R" notation for the PLR group.
22. See Armstrong 15–18 for more on dihedral groups.
23. See Armstrong 32–36 for an expanded discussion of group isomorphisms.
24. For more on defining relations see Artin 219–23.
25. A subgroup H of a group G is the group H composed of a subset of elements from G which uses the binary composition from G.
26. Alternatively, one can say that scrutinizing a group in isolation involves group actions of a G on an S, but where S = G, that is, where the space consists of elements of the group.
27. See Clampitt 1998 which compares the analytical implications of the two systems in an analysis of a passage from *Parsifal*.
28. In this line of thinking we incorporate a conclusion from earlier in this paper: *the elements of an STRANS group are transpositions by the intervals of the corresponding GIS*. In this context "transposition" refers to transposition by GIS intervals, which, as Figure 10 illustrates, is not necessarily the same thing as pitch-class transposition.
29. For more on dual groups see Fiore and Satyendra 2005.
30. It can be proved that the elements of COMM are precisely the operations that commute with SIMP. By this we know that COMM contains all the commuting elements g′ that satisfy the equality $(g * g') = (g' * g)$ for all g in SIMP.
31. PETEY can also be described as the group generated by COMM and SIMP, since any combination of operators can be reduced to the form PT. This reduction is possible because any P commutes with any T.
32. PETEY from $STRANS_1$ and $STRANS_2$ contains 32 operations. Note that both groups have eight elements. By a theorem of group theory, we conclude that the size of the group is $(8 \times 8)/2 = 32$. See Armstrong 169.

33. The synonymity of commutativity with interval preservation is given in Lewin 1995 (101) and proven in Fiore and Satyendra 2005, Theorem 2.1.
34. Thanks go to David Clampitt for pointing out this procedure to me.
35. The group generated by TNSPS and INVS is the set-theoretic union of TNSPS and INVS since the product of a transposition and an inversion is an inversion.
36. This view of transformational networks as existing outside of the temporal flow of a piece accords with Lewin's statement that "music theory attempts to describe the ways in which, given a certain body of literature, composers and listeners appear to have accepted sound as conceptually structured, categorically prior to any one specific piece." (Lewin 1991, 112).
37. While this section characterizes Lewin's theory in terms of his own statements, a theory of course can be fruitful in ways that go beyond or even against the intentions of its author. Accordingly, this discussion makes no claim that we should limit the possibility of Lewin's theory through intentional arguments.
38. Lewin situates his work within the history of interval theory in "Appendix A: Melodic and Harmonic GIS Structures; Some Notes on the History of Tonal Theory" (GMIT 245–50).
39. Lewin remarks "it is unfair to demand of a music theory that it always address our sonic intuitions faithfully in all potential musical contexts under all circumstances. It is enough to ask that the theory do so in a sufficient number of contexts and circumstances" (GMIT 85).
40. In Lewin's words, "One can only demand that a preponderance of its [a theory's] true statements be *potentially* meaningful in sufficiently developed and extended perceptual contexts" (GMIT 87). This view is not peculiar to Lewin but is reflected in much analytical work. Cone, for instance, writes "The 'interesting facts' about such a work are not those that are simply true but those that are relevant to our perceptions" (Cone 5).
41. Though there is no consensus on what comprises a scientific theory the covering law model is the most discussed example of a scientific theory (Lycan 411). A modified version of this model was suggested for music theory by Brown and Dempster (1990). In the covering-law model, valid inferences from data are meaningful without the need for interpretive adjudication. In their article Brown and Dempster argue that a proper music theory should have the kind of predictive and explanatory power of a deductive-nomological scientific theory.
42. The idea that mathematics supports scientific music theorizing is associated with Milton Babbitt who advocated that music theories should be scientific. Given the formative role of Babbitt's work in the history of recent music theory, the fact that both Lewin and Babbitt draw on the language of mathematics in general and on group theory in particular begs the question of whether Lewin's work aims to be in some sense "scientific." While this article argues against a scientific view of Lewin's theory, Babbitt's emphasis on precision in theoretical language resembles Lewin's. For more on the linguistic turn in Babbitt's scientific view of theory, see Guck 1997.
43. John Rahn notes "We must not take the presence of formal language as a stigma or sign pointing to a belief in D-N or in certain paradigms for scientific research, or indeed to anything other than a desire to express ideas formally—though one might venture further to hypothesize that the author of formal language might

believe that in the context of his text, the use of formal language serves communication well" (Rahn 1989, 146).

44. The mathematical notion of simple transitivity, for instance, guarantees that we will have no ambiguous interval names (the interval from s to t will have exactly one value).
45. Kevin Korsyn remarks "Lewin's listening subject, I submit, also lives in ironic mode, constantly aware of the possibility of multiple redescriptions of its own experience" (Korsyn 173). Korsyn's comment refers to the study of the temporal experience of music in Lewin 1986, which considers the ongoing reinterpretation of passing musical events in light of subsequently heard events.
46. As Lewin puts it, "we do not really have one intuition of something called 'musical space.' Instead, we intuit several or many musical spaces at once." (GMIT 250). When the relationships in each space support it, GIS theory allows for the integration of multiple intervallic intuitions into a single system through various means such as "direct product groups" (GMIT 37–46).

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