Norman Carey, “Introduction,” from *Distribution Modulo 1 and Musical Scales*, PhD. Dissertation, University of Rochester, 1998, pp. 2–10.

A. Introduction

Over the last twenty-five years or so, structural properties of musical scales have become the focus of a field of music theory that has sometimes been called “diatonic set theory.” The name is misleading on several counts. Invoking the phrase “set theory” makes an association with the inappropriate term “set theory” for atonal theory. (For an evaluation of this misuse of terminology, see Morris 1994.) While atonal theory has provided a basis and a language for analysis of atonal music, its diatonic counterpart has not been focused on analytical methods for its own repertoire. And, what is its repertoire? Recent scale theorists have proposed a variety of generalized scale systems, modeling a significant number of scales in use in the musical systems of many cultures, and so the potential repertoire for diatonic theory is vast. While there has been some analytical work in diatonic theory involving tonal music, so far these attempts have been confined to the study of isolated musical fragments, and have avoided more far-reaching questions such as form and tonal structure.

The “diatonic” aspect of the title indicates the fact that the “white-key” diatonic scale has served as a point of departure for more general scale theories. This word is problematic as well, belying possible bias towards a view of the diatonic as somehow normative. There is ample reason for substituting the more neutral expression “scale theory,” however, even with this, we would not eliminate ethnocentric bias altogether, since not all cultures structure pitch space scalarly. Clearly, though, “scale theory” is the general term here, while the term “diatonic theory” will be used

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in this dissertation to refer to the field of recent theory that has studied the structural properties of the diatonic and other scales.[1]

In one sense, there is no well-defined way to differentiate between “scales” and “pitch-class sets.” For some theorists, Jay Rahn in particular, being able to establish criteria by which to make this distinction is an important concern. The notion of “scale” often carries with it a constellation of related notions, those of “scale step,” “mode,” “ambitus,” “tonality,” etc., not to mention the idea of a temporal unfolding of a series of ascending or descending pitches. Provisionally it may be useful to think of scales as sets of pitches within a single “octave.”

Each diatonic theory deems certain properties of scales to be essential and invariant and others to be less essential and mutable. The former are maintained as constants, while the latter become variable parameters of the theory. This dissertation focuses on one particular diatonic theory, that of the *well-formed scale*. The notion of the well-formed scale was first delineated in “Aspects of well-formed scales,” by the present author and David Clampitt. (Carey and Clampitt 1989) Section D, below, consists of a summation of some of the theorists whose works can be seen as precursors for more comprehensive diatonic theories. Section E examines three of these theories in depth, those of Balzano, Agmon, and Clough. A synopsis of the theory of

[1]In a larger sense, the underlying idea of much of this work is that of cycles, and interactions (or interferences) between cycles. By taking this broader approach, the applications are similarly broadened and easily extended into other musical domains, rhythm being the most obvious. Some of these will be examined in Chapter IV. I will resist this possibility, however, hoping that we will gain in depth what we lose in breadth.

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well-formed scales begins in Chapter II. First, we specify those properties of musical scales that are the substance of scale theories in general.

B. Foundational and Structural Properties

The study of musical scales is centered upon two primary components, what we will call here the *foundational* and the *structural*. The foundational component of a musical scale is a set of basic intervals. In scale systems generally, certain intervals function as "primitives," playing generational and/or referential roles in the resulting systems. The rationales proffered to justify the selection of one set of foundational intervals over another are diverse. Among other reasons, foundational intervals have been preferred for their acoustical properties (such as consonance as identified with the lack, or minimization, of beating), for their position in the overtone series, for their association with small number ratios, and so on. The arrangement and organization of the foundational intervals in a scale constitute the structural component. As illustrated below, the structure of a scale may be modeled by the pattern of step intervals it maintains.

The foundational and structural components of scales are interdependent, but not identical. Having decided, for whatever reasons, on a set of foundational intervals, the structural problem of organization and arrangement still remains. Various tunings of the diatonic scale bring out some relationships between the foundational and structural components. In Pythagorean tuning, there are two foundational intervals, the octave (2:1) and the perfect fifth (3:2). Seven pitch classes are generated by six perfect fifths, and reduced modulo the octave. The structure of the scale is represented by the sequence of its step intervals:

(1.1) W W H W W W H

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where W stands for the interval of the Pythagorean major second, 9:8, and H stands for the minor second, 256:243. If, instead, the foundational pair is the octave and a tempered fifth, say $2^{\frac{7}{12}}$, the fifth of equal temperament, the resulting seven-note scale can still be represented by sequence 1.1. Since the parameters W and H can take on any positive value, it will generally be to our advantage to use letters such as "A" and "B," which are free of tuning implications in representing such a sequence. Thus, 1.1 would appear:

(1.2) A A B A A A B

Each such sequence, then, defines a class of scales whose members differ from each other with respect to tuning only, that is, with the choice of foundational intervals. Blackwood (1985) examines the class represented by sequence 1.2, imposing on it two additional constraints: 1) the foundational intervals are the invariant octave, and a variable fifth; 2) the range of allowable variation in the fifth is limited to cases in which the resulting step interval represented by 'A' ('W' in 1.1) is larger than the one represented by 'B' ('H').

Quite a different situation prevails in the diatonic scale under just intonation. We would need to posit three foundational intervals; the octave, the perfect fifth and the major third (5:4). A method of producing the just scale runs as follows: From the fifth and the third build a substructure, the major triad (4:5:6). Three triads joined by perfect fifths produce all of the pitch classes of the scale. In Example I-1, triads are found in each row, while the adjacent elements in each column are joined by fifths. The proportion 4:6:9 is formed of a pair of compounded 2:3 ratios (4:6 = 6:9 = 2:3). The octave serves as a modular frame. Multiplying or dividing by 2 as needed brings all of the notes into a single octave.

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[Example I-1]

Unlike Pythagorean tuning or equal temperament, there are two different kinds of whole steps in the just scale: a larger, 9:8 and a smaller, 10:9. There is one kind of half step, 16:15. If we let A stand for the larger whole step, B for the smaller whole step, and C for the half step, the structure of the just scale is represented by the sequence 1.3:[2]

(1.3) A B C A B A C

The two derivations of the diatonic scale (whose structures are shown in 1.2 and 1.3) conflict, and this has far-reaching implications for diatonic music. Essentially, the conflict sets fifths against thirds as foundational intervals. Pythagorean tuning maintains overtone perfect fifths, which results in unacceptably wide major thirds according to just intonation: conversely, maintaining overtone major thirds results in at least one fifth, namely the one between D and A, that is unacceptable to Pythagorean tuning. Fifth generation is more melodic in orientation, whereas just intonation is more harmonic. Generation by fifth provides support for the medieval modal system, with its concomitant concern with the placement of half steps, and, in fact, medieval modal theory is based on Pythagorean tuning. Fifth generation is also allied with our system of keys, key signatures, and modulations. The primacy of the triad in just intonation is naturally related to harmonic concerns: on the other hand, the just scale is

[2]The same pattern arises in the *ma-grama* form of the Indian scale, where A, B, and C represent intervals of 4, 3, and 2 śrutis respectively. See Harold Powers in *New Grove Dictionary*, "India" §III.1.ii. (in particular, Table 2).

notoriously deficient with regard to modulation.[3] Beginning in the 17th century, the gradual ascendance of 12-tone equal temperament imposed an uneasy balance on the conflicting foundations of the diatonic scale. (See Barbour 1933.)

While foundational concerns figure prominently in older accounts of the diatonic scale, the focus here is on the structural aspects of the diatonic and other scales. Tuning will be considered principally to observe how it affects the structure of scales, as revealed in the number of their pitch classes, the distribution and sequence of step interval types, the relationships between interval cycles, and so on.

The cognitive importance of structural properties is described by Gerald Bal-

zano:

> [W]hen we look at the just, Pythagorean, and equaltempered tunings of our familiar major scale, it is evident that there is an important sense in which they work in substantially the same way … The ratios, which are different in all three tuning schemes, do not really address this fundamental commonality. This fact alone should have told us that ratios per se are not the basic descriptors of what goes on in a piece of diatonic … music; that ratios … may even be an inappropriate level of description for pitch systems. Now there is no question that without ratios, we would have never discovered and refined the splendid [diatonic/chromatic system]. But let us not confuse historical importance with perceptual importance. It may well be that the group-

[3]In Just Intonation, modulating from C to G is already problematic since the dominant chord of G contains an out of tune fifth between D and A, 40:27. (An overtone perfect fifth above 27 is 41.5).

> theoretic properties … were the more perceptually important all along. (Balzano 1980, 84.)

In this excerpt, Balzano's "group-theoretic properties" are examples of our "structural properties." While we have shown above that the just scale is structurally different from the other two tunings mentioned, the point is nevertheless made: Structural properties are "perceptually, " hence musically important.

C. Definitions

In order to begin our survey of recent diatonic theory, we provide the following definitions:

1) *Interval of periodicity*. An interval whose two pitches are functionally equivalent. Normally, the octave is the interval of periodicity. Informally, "octave" in double quotes means interval of periodicity.

2) A *scale* is a set of pitches, ordered by frequency, which span the interval of periodicity. A scale is understood as potentially extending infinitely, replicating itself by "octaves" throughout pitch space. A scale is defined as unique up to rotation.

3) The *cardinality* of a scale with *N* notes is *N*. An *N*-note scale can be represented by the *N* pitch classes 0, 1, …, *N* – 1. (N.B: Pitch classes given in order of generation, not scalar order.)

4) Each of the *N* distinct pitch classes of a scale is the point of origin of one of its *modes*. Each of the modes is a representation of the scale.

5) A *step* is an interval that connects two adjacent notes of a scale.

6) An *equal-interval scale* subdivides its interval of periodicity into equal steps.

7) A scale is said to be *generated* if all of its pitch classes are formed by successive multiples (reduced modulo the interval of periodicity) of a single interval (called a *generator*). The interval of the perfect fifth provides an archetype for this function. Any equal-interval scale has at least one generator, namely, the step. When the cardinality of a generated scale is *N* the generator itself will appear *N* – 1 times. If, however, the generated scale is also an equal-interval scale, the generator appears *N* times.[4]

8) A *generalized comma* for a generated scale of *N* notes is the smallest interval that can be formed between pitch class 0 and pitch class *N*. (N.B: Pitch class *N* does not belong to the *N*-note scale, 0, 1, …, *N* – 1.) For example, we compute the generalized comma for the pentatonic scale, which consists of the five pitch classes F(0), C(1), G(2), D(3), A(4). The smallest interval between the next pitch class, E(5), and the origin, F(0), is a minor second. Thus, the minor second is the generalized comma for the pentatonic scale. The minor second is not found in the pentatonic scale, and as a general rule, the generalized comma is not found in the associated *N*-

[4]Barbour's "regular temperament" is an example of a generated scale. "[Regular temperament is] a temperament in which all of the fifths save one are of the same size, such as the Pythagorean tuning or the meantone temperament. (Equal temperament, with all fifths equal, is also a regular temperament, and so are [any other equal interval scales.])" (Barbour 19~~51, xi.) The notion also calls t~~o mind Curt Sachs's distinction that scales are built either according to a "cyclic" principle or a "divisive" one. (Sachs 1943). Generated scales are, in his sense, cyclic.

note scale. There are, as we will see, advantages when the generalized comma of a generated scale is "small," in some definable sense.

9) A scale is said to be *well-formed* if a) it is generated; and b) the number of steps in the generator is constant in each of its *N* – 1 appearances. We can easily verify that the diatonic scale is well-formed: The perfect fifth is a generator of the diatonic scale, and the word "fifth" itself testifies to the fact that every time it appears in the scale it is subdivided into four step intervals. As we will see, the generalized comma of a well-formed scale is always smaller than at least one of the step intervals.

10) An equal-interval scale is called a *degenerate well-formed scale*. The generalized comma of a degenerate well-formed scale is unison.[5]