Defining Modular Transformations

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I. BARTÓK'S TRANSFORMATION AND ITS GENERALIZATION

In a lecture at Harvard University in 1943, Béla Bartók acknowledged his discovery and use of a transformation that maps musical entities from chromatic to diatonic collections:

> Working [with] chromatic degrees gave me [an] idea which led to the use of a new device. This consists of the change of the chromatic degrees into diatonic degrees. In other words, the succession of chromatic degrees is extended by leveling them over a diatonic terrain.
>
> You know very well the extension of themes in their values called augmentation, and their compression in value called diminution. These devices are very well known. . . . Now, this new device could be called "extension in range" of a theme. For the extension we have the liberty to choose any diatonic scale or mode.[1]

Examples 1–3 show instances of this transformation in Bartók's music. Example 1, from *Music for Strings, Percussion, and Celesta,* shows how the beginning of the chromatic theme of the first movement, <A B♭ C♯ C B>, maps onto the beginning of the diatonic (Lydian) theme in m. 204 of the fourth movement, <C D G F♯ E>. That is, scale-degree 1 from the chromatic collection maps onto scale-degree 1 from the diatonic, scale-degree 2 maps onto scale-degree 2, and so forth, as shown by the scale-degree numbers beneath the example. Examples 2 and 3 are taken from *Mikrokosmos.* Example 2 shows how piece No. 64a maps onto No. 64b by a transformation that compresses the diatonic music into a chromatic space, while Example 3 shows how the theme from the beginning of piece No. 112 maps onto the theme at the beginning of its second section (marked *un poco meno mosso*) by a transformation that compresses the diatonic music into a chromatic space. Numerals in parentheses below the staves in Example 3 represent scale degrees that do not participate.[2]

But Bartók's transformation need not be limited to mappings between diatonic and chromatic spaces; one can find examples of mappings to and from other modular spaces in Bartók's own music. Example 4 is taken from the first movement of his Fourth String Quartet, and shows how the chromatic first theme in m. 7, <B C D♭ C B B♭>, maps onto the octatonic segment in the first violin part of m. 158, <E♭ F G♭ F E♭ D>. This article formalizes

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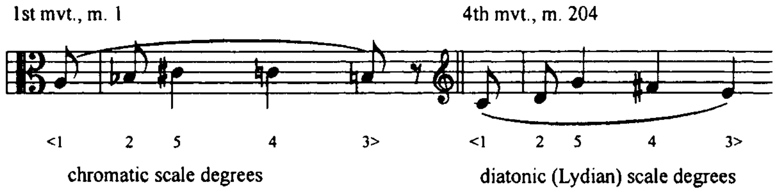
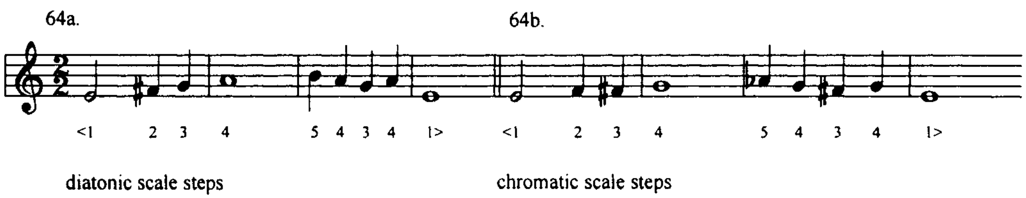
[1] Béla Bartók, "Harvard Lectures," in *Béla Bartók Essays,* ed. Benjamin Suchoff (New York: St. Martin's Press, 1976), 381. Wayne Alpern has also explored this remodularization technique in Bartók's music from a transformational perspective in "Bartók's Compositional Process: 'Extension in Range' as a Progressive Contour Transformation," paper presented at the annual conference of Music Theory Midwest, 15 May 1998. Examples 1, 2, 3, and 12 are taken from Alpern's paper. Alpern's work generalizes this operation to non-modularized progressive contour transformations in Bartók as well as Berg, whereas this article retains an exclusively modular perspective.

[2] The terms *modular space* and *modular system* will be defined here as synonyms referring to specified partitionings of the octave within the larger context of an equal-tempered harmonic system.

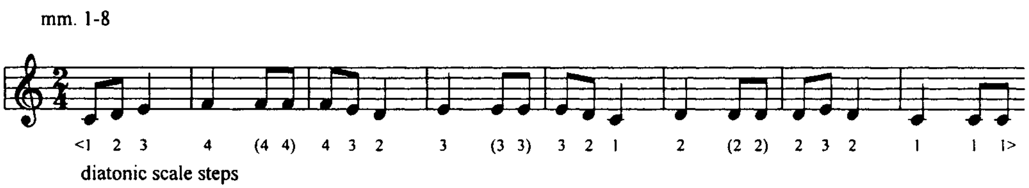
Example 1. Bartók, *Music for Strings, Percussion, and Celesta,* I and IV; transformation from chromatic to diatonic

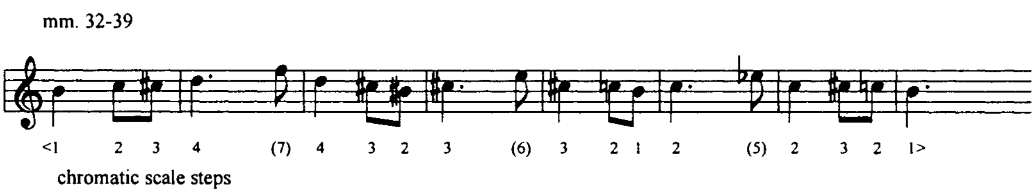


Example 2. Bartók, No. 64a and 64b from *Mikrokosmos,* mm. 1–4 of each; transformation from diatonic to chromatic

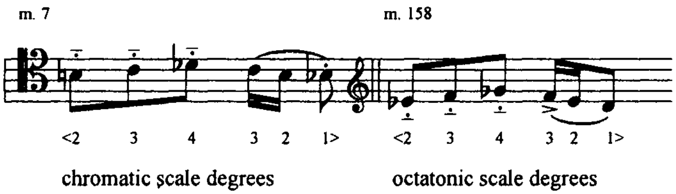


Example 3. Bartók, No. 112 from *Mikrokosmos;* transformation from diatonic to chromatic





Example 4. Bartók, String Quartet No. 4, I; transformation from chromatic to octatonic



Bartók's transformation, and generalizes it to map musical entities to and from any one of five different modular spaces: chromatic, octatonic, diatonic, whole-tone, and pentatonic.[3]

It is possible to discuss Bartók's transformation in a more systematic way. Before doing so, however, it is necessary to replace the term "scale degree," which has potentially misleading diatonic and tonal connotations, with the term "step class."[4] A step class is defined here as a numbered position within a modular system; octave equivalence is assumed. Step classes are numbered 0 to n (n being equal to the cardinality of the modulus minus 1).[5] Thus each step class in a mod7 (diatonic) space, for example, will be equal to the corresponding scale-degree minus 1 (e.g., scale-degree 4 is equivalent to step-class 3).

Let us define MODTRANS (x, y, z) as a transformation that maps each step class of a musical entity in modular system *x* onto a corresponding step class in modular system *y*, where *z* represents the "point of synchronization," the pitch class in the starting modulus that is interpreted as step-class 0. From this point of synchronization, one may then construct a table of mappings from the starting modulus to the destination modulus and use the table to map pitch classes from the first musical entity to the second one. Example 5 displays all of the possible rotations for the mod12, mod8, mod7, mod6, and mod5 systems considered here in integer notation, arranged so that one may easily find their corresponding step classes; the table can thus be used to find the mappings for any MODTRANS operation occurring between the five systems. The integer notation in Example 5 sets the point of synchronization equal to 0, and the remaining pitch classes in each modular system receive a value equal to their distance in semitones above the point of synchronization.

Because mappings among systems partitioned into unequal steps are variable depending on where in the system the point of

[3] I have restricted my focus to these spaces because these are the most important subdivisions of the octave into collections of twelve, eight, seven, six, and five notes, respectively. Though many other seven-note collections could also be considered, I am understanding those to be derivations from a diatonic norm. I exclude harmonic and melodic minor, for example, as derivations from the natural form. It will be easy for the reader to adapt my methods to such derived spaces, but it would be overly cumbersome to include them all here.

There is a rich literature on the properties of modular systems. See, for example, Milton Babbitt, "The Structure and Function of Music Theory I," *College Music Symposium* 5 (1965): 49–60; Hubert Howe, "Some Combinational Properties of Pitch Structures," *Perspectives of New Music* 4/1 (1965): 45–61; Carlton Gamer, "Some Combinational Resources of Equal-Tempered Systems," *Journal of Music Theory* 11/1 (1967): 32–59; Robert Cogan and Pozzi Escot, *Sonic Design* (Englewood Cliffs, N.J.: Prentice Hall, 1976); Jay Rahn, "Some Recurrent Features of Scales," *In Theory Only* 2/11–12 (1977): 43–52; Richmond Browne, "Tonal Implications of the Diatonic Set," *In Theory Only* 5/6–7 (1981): 3–21; Robert Gauldin, "The Cycle-7 Complex: Relations of Diatonic Set Theory to the Evolution of Ancient Tonal Systems," *Music Theory Spectrum* 5 (1983): 39–55; John Clough and Gerald Myerson, "Variety and Multiplicity in Diatonic Systems," *Journal of Music Theory* 29/2 (1985): 249–70; Norman Carey and David Clampitt, "Aspects of Well-Formed Scales," *Music Theory Spectrum* 11/2 (1989): 187–206; and Jay Rahn, "Coordination of Interval Sizes in Seven-Tone Collections," *Journal of Music Theory* 35/1 (1991): 33–60.

[4] Stephen Dembski used the term "step class" in "Steps and Skips from Content and Order: Aspects of a Generalized Step-Class System," paper presented at the annual meeting of the Society for Music Theory, Baltimore, 5 November 1988.

[5] Because "step class" is defined here as an order position within a modular system, there is a danger that it might be confused with the term "order position." This article advocates using "order position" to refer to ordering in a context that is specific to a musical work, such as a twelve-tone row, and reserving the use of "step class" for orderings of the five modular systems discussed here.

Example 5. Possible rotations of the mod12, mod8, mod7, mod6, and mod5 systems

modulus	label	Step Classes											
		0	1	2	3	4	5	6	7	8	9	10	11
chromatic	12	0	1	2	3	4	5	6	7	8	9	10	11
octatonic	8^1	0	1	3	4	6	7	9	10				
-	8^2	0	2	3	5	6	8	9	11				
diatonic	7^1	0	2	4	5	7	9	11					
-	7^2	0	2	3	5	7	9	10					
-	7^3	0	1	3	5	7	8	10					
-	7^4	0	2	4	6	7	9	11					
-	7^5	0	2	4	5	7	9	10					
-	7^6	0	2	3	5	7	8	10					
-	7^7	0	1	3	5	6	8	10					
whole-tone	6	0	2	4	6	8	10						
pentatonic	5^1	0	2	4	7	9							
-	5^2	0	2	5	7	10							
-	5^3	0	3	5	8	10							
-	5^4	0	2	5	7	9							
-	5^5	0	3	5	7	10							

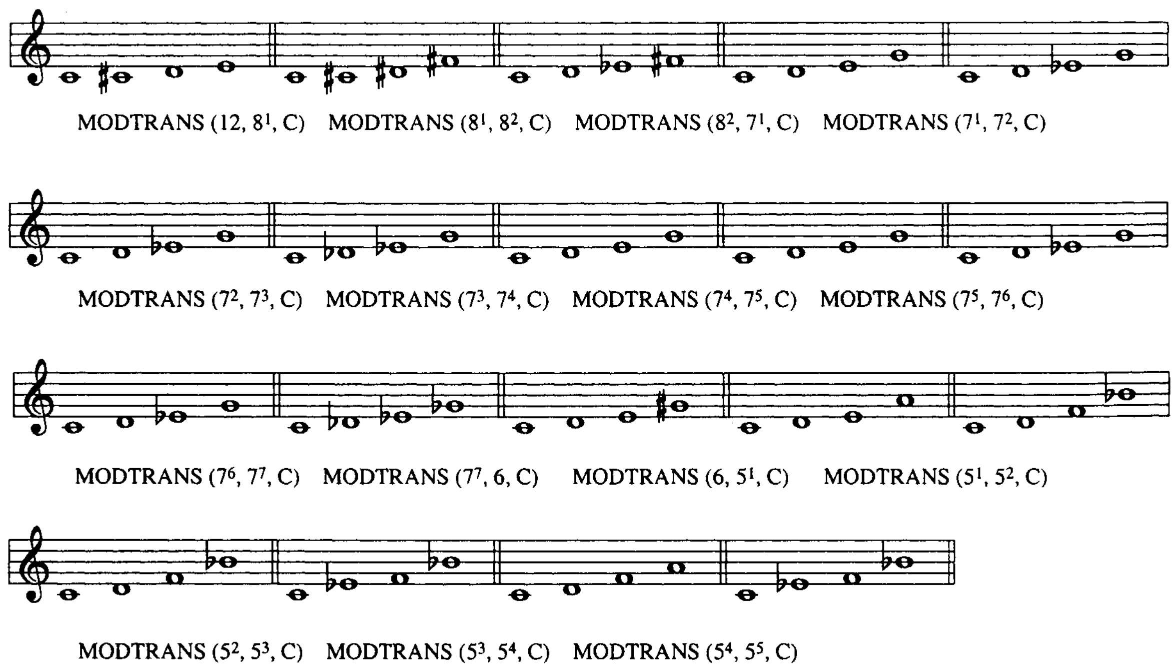
synchronization lies, Example 5 lists each distinct rotation of those with unequal steps—the octatonic, diatonic, and pentatonic—but lists those with all equal steps—the chromatic and whole-tone—only once each. An octatonic system beginning with a half step is represented as 8^1; an octatonic beginning with a whole step is 8^2. The diatonic system is represented in seven different ways, one for each of the seven diatonic modes: Ionian is 7^1, Dorian is 7^2, Phrygian is 7^3, and so on. The five rotations of the pentatonic scale are labeled as follows: 5^1 = <02479>, 5^2 = <0257T>, 5^3 = <0358T>, 5^4 = <02579>, and 5^5 = <0357T>.

Example 6 illustrates how this table works by transforming step-class segment <0124> through each rotation of the modular spaces in Example 5 (proceeding from top to bottom). First, MODTRANS (12, 8^1, C) maps <C C♯ D E> of chromatic space to <C C♯ D♯ F♯> of the first octatonic collection, thereby connecting the first two rows on the table of Example 5; then MODTRANS (8^1, 8^2, C) maps <C C♯ D♯ F♯> to <C D E♭ F♯>, connecting the two octatonic collections on the second and third rows, and so forth. The choice of step-class 0 as the point of synchronization is arbitrary and non-prejudicial.

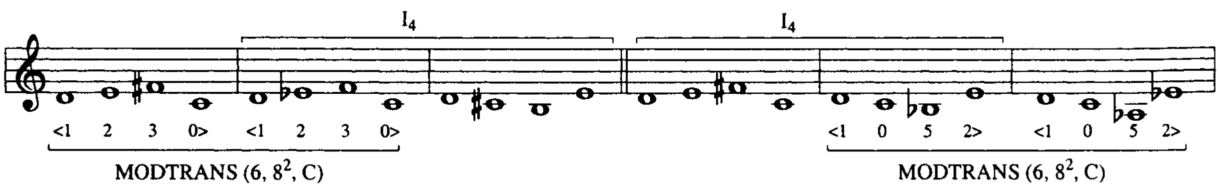
In actual musical applications one must keep in mind that MODTRANS often occurs in combination with transposition; that is the case in Examples 1, 3, and 4. Thus, in using Example 5 to interpret Example 4, for instance, we set its lowest note, B♭, equal to 0 and map the first four mod12 step classes in m. 7—(0123), or (B♭ B C D♭)—onto the first four mod8^1 step classes listed below it—(0134), or (B♭ B C♯ D)—and then transpose the result up a major third to (D E♭ F G♭). The full label for this combination of transformations includes both MODTRANS (12, 8^1, B♭) and T_{+4}. While MODTRANS is commutative with respect to mod12 transposition, it is not commutative with respect to mod12 inversion.[6] Example 7 illustrates. The whole-tone segment <D E F♯ C> is transformed by MODTRANS (6, 8^2, C) to yield <D E♭ F C>, which is inverted about a D axis to produce <D C♯ B E>. But reversing the order of operations changes the result: <D E F♯ C> inverted about D produces <D C B♭ E>, which maps via MODTRANS (6, 8^2, C) to <D C A♭ E♭>. (Note that while mod12 transposition preserves the original succession of step classes, mod12 inversion does not.) In order to avoid problems related to

[6]The commutativity of the MODTRANS operation with respect to transposition may be proven as follows: given the operations MODTRANS (x, y, z) and T_n, let j^1 equal the ordered pc interval from the point of synchronization to step-class j in modular system x, let j^2 equal the ordered pc interval from the point of synchronization to step-class j in modular system y, let $i = j^2 - j^1$, let pitch-class k represent step-class j in the musical entity to be transformed, let m represent the pitch-class onto which k is mapped under MODTRANS (x, y, z) followed by T_n, and let p represent the pitch-class onto which k is mapped under T_n followed by MODTRANS (x, y, z). PROOF: $m = ((k + i) + n)$, $p = ((k + n) + i)$, therefore $m = p$.

Example 6. Successive MODTRANS operations applied to (0124)



Example 7. MODTRANS followed by inversion vs. inversion followed by MODTRANS



the limited commutativity of the MODTRANS operation, we will establish an ordering convention in which the MODTRANS operation will always be applied *before* the other operations.[7]

With respect to Example 7, some might advocate inverting <D E♭ F C> within the octatonic collection—employing mod8 inversion—and inverting <D E F♯ C> within the whole-tone collection—using mod6 inversion—rather than applying mod12 inversion to both of them. But employing inversions in other modular spaces offers no analytic or methodological advantages. In Example 7, for instance, a mod8 inversion-about-D of <D E♭ F C> would yield <D C B E♭>, rather than <D C♯ B E>. A mod6 inversion-about-D of <D E F♯ C> would produce the same result as the mod12 inversion. The results are no more revealing than when mod12 is used exclusively; inversion is still not commutative. Since mod-12 operations are familiar, it makes sense to retain them.

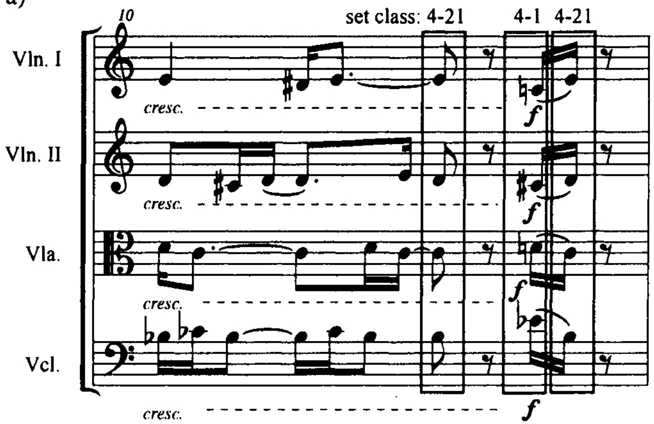
Although we have only examined melodic segments in the examples thus far, the MODTRANS operation can just as easily transform simultaneities. Example 8a gives m. 10 from the first movement of Bartók's Fourth String Quartet. Example 8b is an interpretation of the underlying voice-leading of the passage: notes in the first and third columns represent notes in the music, while notes in the second column illustrate transformational pathways. The whole-tone tetrachord, set class 4-21 (0246), is related to the chromatic tetrachord, set class 4-1 (0123), by MODTRANS (6, 12, B♭) and I_1. Musical context justifies interpreting the transformation as I_1 rather than the other possibility T_2: the inversional mappings occur within the respective instrumental lines, as Example 8b illustrates.

If a set class is a subset of two or more moduli, it may be represented by any of those moduli. Example 9 shows that the motion from <C D E F♯> to <C C♯ D D♯> could be labeled as either

[7]The convention is necessary even though MODTRANS *is* commutative with respect to transposition, and thus the order of operations when combining MODTRANS and transposition is irrelevant.

Example 8. Bartók, String Quartet No. 4, I, m. 10

a)



b)

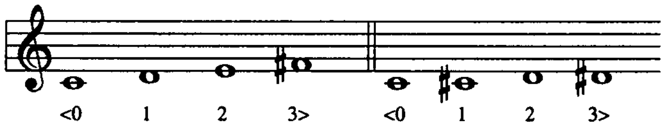
set class:	4-21		4-1
Vln. I	E →	(C♯) →	C
Vln. II	D →	(C) →	C♯
Vla.	C →	(B) →	D
Vcl.	B♭ →	(B♭) →	E♭

MODTRANS (6, 12, B♭)

I_1

MODTRANS (6, 12, C) or as MODTRANS (7^4, 12, C), because <C D E F♯> is a subset of both the whole-tone scale and the fourth mode of the major scale. Again musical context will guide the interpretation. If <C D E F♯> appears in a musical environment composed mainly of whole-tone subsets, one would prefer a whole-tone interpretation, while if it is part of a passage drawn from the notes of a G major scale, a diatonic interpretation would

Example 9. Multiple MODTRANS interpretations of a single transformation



MODTRANS (6, 12, C)
or
MODTRANS (7^4, 12, C)

be more appropriate. In a musical context that is constantly shifting between different modular spaces, regardless of whether they were chromatic, octatonic, diatonic, whole-tone, or pentatonic, one might interpret it as whole-tone, based on the fact that its set class, (0246), is imbricated six times in the whole-tone collection and is thus more indicative of that collection than of the diatonic one, in which it is embedded only once.

Given that the modulus of a musical entity is often subject to more than one interpretation, the reader might wonder whether the MODTRANS operation is capable of linking any two sets of the same cardinality. This is not the case, because of the limited number of modular contexts under consideration. Example 10 shows how trichords are interrelated. It lists the number of MODTRANS mappings connecting any two of the twelve possible trichord types between pentatonic, whole-tone, diatonic, octatonic, and chromatic spaces. We see here that (026) is capable of mapping onto the greatest number of trichord types, nine total, while (012) is the most limited in this capacity, only capable of mapping onto three.

Example 5 does not account for what happens to higher step classes when moving from a larger to a smaller modulus; for instance, there are no mod6 equivalents for step classes 7–12. Two possible ways of handling such mappings will be considered here. The first imagines the destination modulus as a clockface numbered according to its cardinality, and in effect wraps the larger starting modulus around the smaller one that is its destination; let us call this strategy "modular wrap-around," or MODWRAP. Under MODWRAP, if step class x in the starting modulus is equal to or greater than the cardinality of the destination modulus, y, it maps onto step class (x – y) in the destination modulus. For instance, modular wrap-around maps step-class 7 in a mod12 space onto step-class 0 in a mod7 space. This is Bartók's own solution in *Music for Strings, Percussion, and Celesta,* as shown in Example 11. The high E in the chromatic theme from the first movement, circled in the score, maps onto the high C in the diatonic theme from the fourth movement, also circled. The table at the bottom of the example shows the specific mappings and is arranged in the same way as Example 5. Note that the modular space in the fourth movement is not one of the diatonic modes included in Example 5, but is the fourth mode of the ascending melodic minor scale, sometimes called the "acoustic" or "overtone" scale.

Example 11 highlights what could be considered a flaw in the MODWRAP solution: because there are not as many step-classes in the destination modulus as there are in the starting modulus, different pitch classes in the starting modulus must map onto the same pitch class in the destination modulus. Though Bartók has made the motion from one modular space to another quite audible by preserving the rhythm and melodic contour in this instance, these choices are independent of the MODWRAP transformation itself. If these characteristics of the theme were not preserved, the relationship between the two would most likely be obscured by the fact that two different pitch classes in the original theme map onto the same pitch class when it is transformed by MODWRAP.

This situation is inevitable whenever the number of step classes actually present in a musical entity to be transformed exceeds the cardinality of its destination modulus. However, if the number of step classes used in the entity to be transformed is equal to or less than the cardinality of the destination modulus, another solution is available; let us call this strategy "module completion mapping" or MODCOMP. Under MODCOMP, step classes in the source are placed in ascending numerical order and

Example 10. Number of MODTRANS mappings connecting any two trichords, s and t

s \ t	(012)	(013)	(014)	(015)	(016)	(024)	(025)	(026)	(027)	(036)	(037)	(048)
(012)	-	6	0	0	0	5	4	0	0	0	0	0
(013)	6	-	2	4	0	34	30	4	6	0	4	0
(014)	0	2	-	4	6	2	18	8	16	0	4	0
(015)	0	4	4	-	2	2	28	12	12	0	8	0
(016)	0	8	8	0	-	9	2	12	30	0	12	0
(024)	4	24	0	0	0	-	16		3	3	8	1
(025)	0	10	10	20	0	0	-	22	24	2	32	0
(026)	0	4	6	12	6	2	28	-	16	4	12	0
(027)	0	0	6	0	15	6	1	10	-	0	24	0
(036)	0	0	0	0	2	3	0	0	14	-	26	3
(037)	0	0	0	0	2	6	2	4	18	20	-	6
(048)	0	0	0	0	0	1	0	0	3	3	8	-

Example 11. Bartók, *Music for Strings, Percussion, and Celesta,* themes from mvts. I and IV

<0 1 4 3 2 0 1 4 5 6 3 2 1 4 7 6 5 3 4 2>

mod12 step classes

<0 1 4 3 2 0 1 4 5 6 3 2 1 4 0 6 5 3 4 2>

mod7 step classes

mvt.	modulus	0	1	2	3	4	5	6	7	8	9	10	11
		step classes											
I	**chromatic**	0	1	2	3	4	5	6	7	8	9	10	11
IV	**diatonic**	0	2	4	6	7	9	10	0	2	4	6	7

N.B. This modular space is not one of the seven diatonic modes included in Example 5, but is the fourth mode of the ascending melodic minor scale, also commonly known as the "acoustic scale" or "overtone scale" (see note 3).

mapped to the numerically ordered step classes of the destination. Suppose we wish to map the mod12 segment X = <G C♯ D E C A>, mod12 step classes <7 1 2 4 0 9>, to the mod6 segment Y = <G♯ D E F♯ C A♯>, mod6 step classes <4 1 2 3 0 5>. The step-class segments are re-ordered as X′ = <0 1 2 4 7 9> and Y′ = <0 1 2 3 4 5>, in preparation for mappings between corresponding locations: 0, the first element in X′, maps to 0, the first element in Y′; 1, the second element in X′, maps to 1, the second element in Y′, 2 in X′ maps to 2 in Y′, 4→3, 7→4, and 9→5. In Example 12a these segments are rewritten in ascending order and X′ is assigned order positions to illustrate the exact correspondences with the step classes of Y′.

Example 12b applies MODWRAP to X to show how the results differ when the two mappings are applied to the same musical entity. MODWRAP (12, 6, C) maps X = <G C♯ D E C A> to Z = <D D E G♯ C F♯>: step-class 7, mod12 (G), maps onto step-class 1, mod6 (D; 7-6 = 1); step-class 1, mod12 (D), also maps onto step-class 1, mod6; 4→4; 0→0; and 9→3. Thus MODWRAP maps both C♯ and G in a mod12 space onto D in a mod6 space, while MODCOMP yields no such duplications.

While Bartók is the only composer to have explicitly described the MODTRANS operation as a compositional technique, one may find examples of its use in works by others, and the operation does not seem to be linked to any one compositional style. In what follows, we will see MODTRANS operations in the music of Debussy, Stravinsky, Schoenberg, and Webern.

David Lewin's analysis of Debussy's *Feux d'artifice* shows the thematic and motivic unity in a diverse musical surface that is constantly shifting between and layering materials that are sometimes pentatonic, sometimes diatonic, and sometimes whole-tone.[8] His Examples 4.14 and 4.15 are reproduced here as Examples 13 and 14. Lewin interprets the theme and its variations taken together as a kind of polyphonic structure that layers an ascent

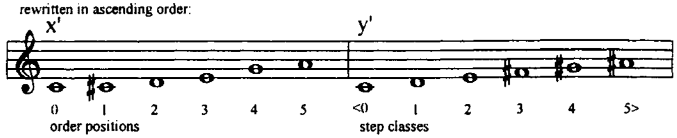
[8]David Lewin, *Musical Form and Transformation: 4 Analytic Essays* (New Haven: Yale University Press, 1993), 97–159.

Example 12. Step-class segment <712409> transformed by MODCOMP and MODWRAP

a. <G C♯ D E C A> transformed by MODCOMP (12, 6, C).

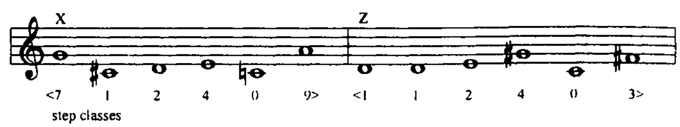
MODCOMP (12, 6, C)

rewritten in ascending order:



MODCOMP (12, 6, C)

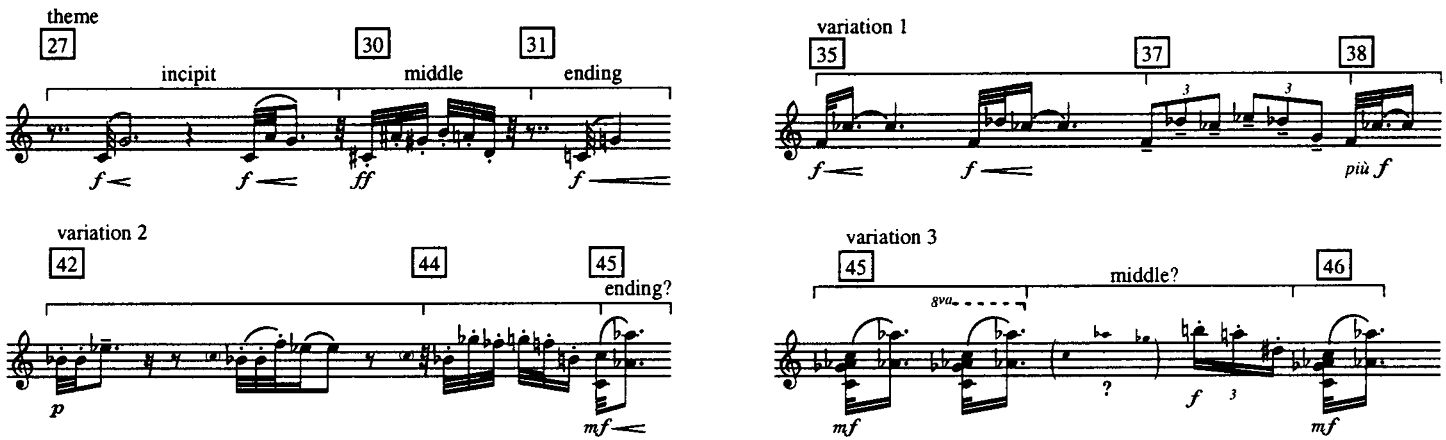
b. <G C♯ D E C A> transformed by MODWRAP (12, 6, C).



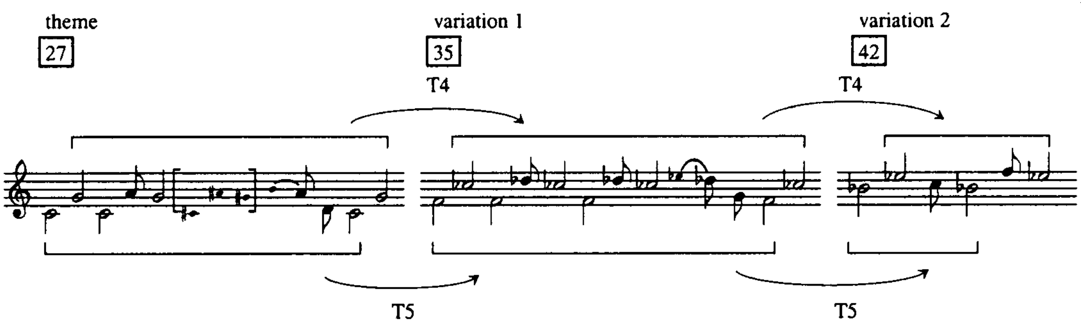
MODWRAP (12, 6, C)
(7 - 6 = 1)
(9 - 6 = 3)

through a whole-tone scale, <G A C♭ D♭ E♭ F>, over an ascent through a pentatonic scale, <C D F G B♭ C>. This interpretation allows him to draw a comparison between the theme and its variations and material from earlier in the piece (mm. 7–8). However, a more harmonic conception of the theme and its variations (i.e., one that explains all of the notes as part of one pitch-class collection) would more readily match our perception of each as a unified whole, rather than suggest an interplay between two contrapuntal voices moving through different modular spaces.

Example 13. Reproduction of Lewin, *Musical Form and Transformation*, Ex. 4.14



Example 14. Reproduction of Lewin, *Musical Form and Transformation*, Ex. 4.15



Example 15 suggests such a harmonic conception. Example 15a gives the theme and its first two variations, while 15b shows the step-by-step process by which each of the transformational pairs is linked. (Whole notes indicate notes in the music; stemless noteheads illustrate transformational pathways.) The incipit of the theme maps onto the incipit of variation 1 by MODTRANS (5^1, 6, C) and T_{+5}, which in turn maps onto the incipit of variation 2 by MODTRANS (6, 7^1, F) and T_{+5}. Thus the incipit of the theme is presented initially in a pentatonic space, and then is varied twice to appear in first a whole-tone space and then a diatonic one. The theme's ending is similarly transformed in variation 1, while in variation 2, another transformational pattern takes hold. As annotated in Example 15a, a pattern of two successive transpositions up a semitone generates both the middle of the theme and the middle and ending of variation 2. The transposition in the middle of variation 1 by T_{+2} may be perceived as a condensation of this pattern.

This perspective embodies many of Lewin's trenchant observations, including the significance of T_{+5} as a pentatonic generator, which guides the lower notes of the theme and its variations along a path that defines a pentatonic collection. However, because Example 15 uses the lowest pitch as the point of synchronization, Lewin's observation about T_{+4} connections between higher notes as a whole-tone generator is lost. This need not be the case—there is no reason why the harmonic conception cannot embrace every aspect of Lewin's commentary. We need only redefine the point of synchronization to be the upper note of the theme's opening interval (as well as those of its subsequent variations). To do this, we insert a fourth argument into the label that indicates the point of synchronization's step class, if it is not considered to be step-class 0. Example 16 illustrates.[9] The transformation mapping the incipit of the theme onto variation 1 is MODTRANS (5^1, 6, G, 3) and T_{+4}, which in turn maps onto the incipit of variation 2 by MODTRANS (6, 7^1, C♭, 3) and T_{+4}.

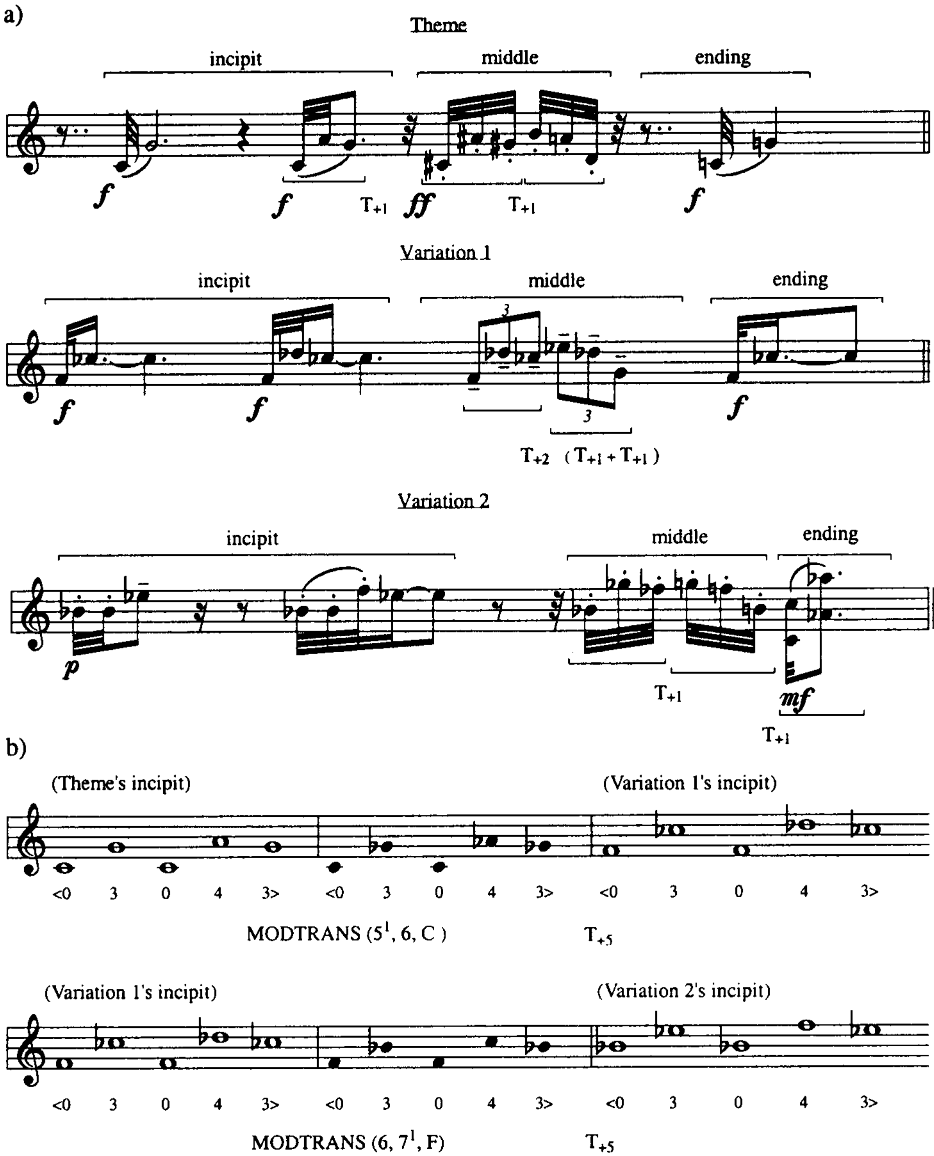
Example 17 shows how the relationship between the two primary motives in a passage from Stravinsky's *Agon* can be understood in terms of a MODTRANS operation. The first motive, the whole-tone segment <C♯ B D♯ E♯> bracketed in Example 17a, accompanies a solo male dancer on stage, and is presented in stretto by the horn and piano parts of mm. 463–468. The second motive, the octatonic segment <D C♯ E F> bracketed in Example 17b, accompanies a solo female dancer, and is presented by the flutes in mm. 473–479. (Although the flutes begin with the motive minus its first note, D, its subsequent occurrences include the D three out of four times.) Example 17c demonstrates a mapping of the first motive onto the second under MODTRANS (6, 8^1, F, 3).[10] Note that the contour of the first motive is not preserved in the second. Though composers such as Bartók and Debussy may preserve melodic contour in their modular transformations—presumably to make the transformation more audible for the listener—MODTRANS always occurs in pitch-class space and does not necessarily preserve contour.

Awareness of the MODTRANS operation here sheds light on one of the movement's many interesting interthematic relationships. The music accompanying the solo male dancer (Example 17a) projects a stereotypical view of masculinity because of its athletic minor-seventh leaps, its loud dynamic level (suggesting strength), and its orchestration (the piano and horn parts together could be characterized as having a bold sound). The music accompanying the solo female dancer (Example 17b), on the other hand, is at a soft dynamic level, is chromatic, and is played by the flutes –the antithesis of the masculine stereotype projected in the A sections. The MODTRANS interpretation suggested here thus recognizes a commonality between two sharply contrasting themes and stereotypes. The similar progression of step classes realized in two

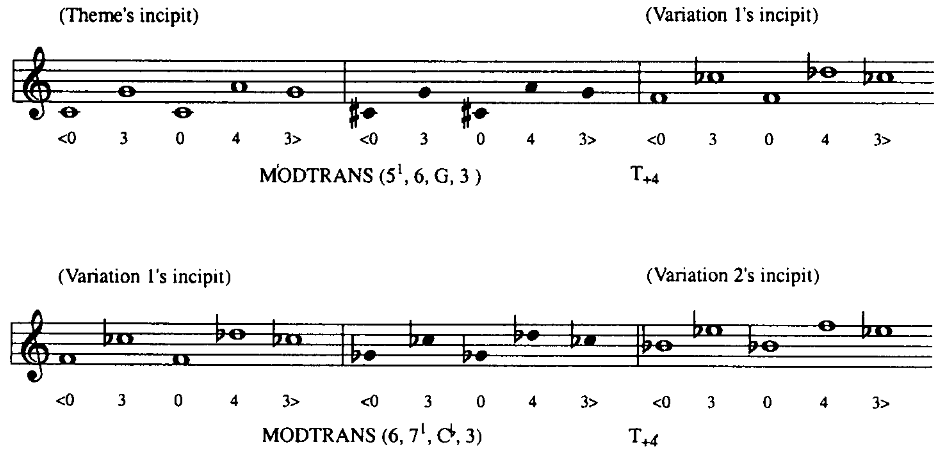
[9]Note, however, that the changing point of synchronization does not affect the labeling of rotations of a given modular space, which is still based on the intervallic sequence above step-class 0.

[10]I wish to thank Gerald Zaritzky for suggesting F as the point of synchronization in this passage.

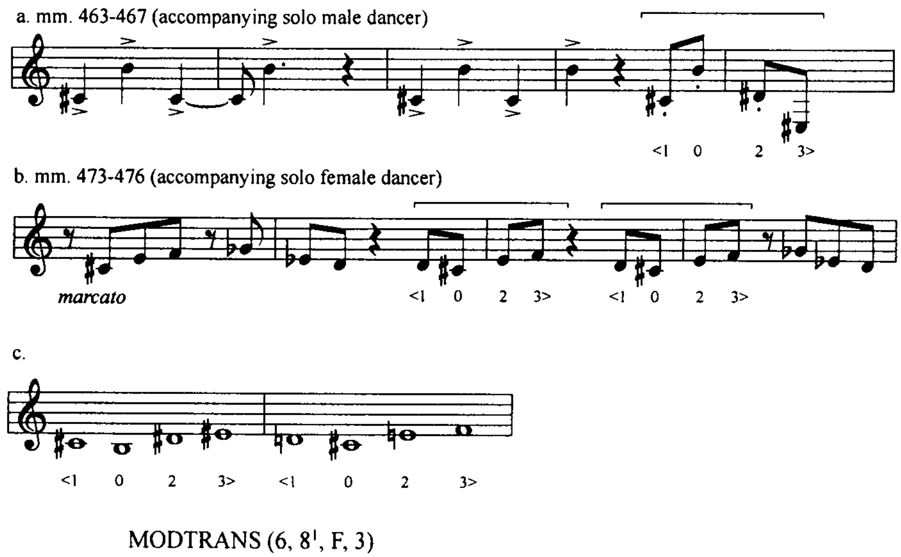
Example 15. MODTRANS mappings in Debussy's *Feux d'artifice*



Example 16. MODTRANS mappings in Debussy's *Feux d'artifice* with step-class 3 as the point of sync



Example 17. Stravinsky, *Agon;* transformation from whole-tone to octatonic in mm. 463–483



different modular spaces may be viewed as a metaphor for a human spirit that transcends gender characterizations.

The step-class segment <2310>, a step-class inversion of the <1023> motive in Example 17 (via inversional mappings 0↔3 and 1↔2), serves to unify the diversity of modular spaces throughout *Agon.* Example 18 shows the interaction of <2310> and <1023> as octatonic step-class segments in an earlier passage from that work.[11] Example 19 finds the <2310> motive occurring in a diatonic space in *Agon*'s opening theme (Ex. 19a) and second important theme in the first movement (Ex. 19b), and in chromatic space in the opening theme of the second movement (Ex. 19c).[12] Examples 19d and 19e demonstrate the mappings.

Thus far, we have seen instances in which the modular spaces suggested by the MODTRANS operation are reinforced (or at least not challenged) by the musical environments of the entities transformed. However, since MODTRANS interpretations are guided by interactions between musical context and abstract properties of the modular spaces, as explained above, such correspondences are not required. Smaller pc sets such as trichords or tetrachords may suggest modular representations based on how indicative they are of one of the modular spaces considered here, even when the larger musical context may not clearly project any one modular space. In the following analysis of the opening of Schoenberg's *Klavierstück* op. 11, no. 1, textures will be marked by the interaction of motive forms presented in constantly shifting modular spaces, some of which may occur simultaneously, as if each motive form asserts its independence partially through the modular space its manifestation suggests.[13]

The piece opens with two melodic trichords: <B G♯ G>, a member of (014), and <A F E>, a member of (015). See Example 20a and the analysis in Example 20b. Although the trichords are not equivalent via mod12 transposition or inversion, they are similar for a number of reasons, including: 1) they are both presented in the same register; 2) they both have the same melodic contour; and 3) they both begin with a leap of a third and end with a motion by step. To explore their relationship we turn to their collectional affiliations and shifting modular spaces.

An analysis of a Schoenberg work that suggests interactions between different modular spaces requires some explanation, given that Schoenberg's preserial music is usually understood in relation to diatonic and chromatic partitionings of the octave. However, we must recognize the potential impact on Schoenberg of the growing interest in symmetrical partitionings of the octave (i.e., in whole-tone and octatonic collections) that began in the latter half of the nineteenth century. The influence of such ideas on his contemporaries, including Debussy, Bartók, and Stravinsky, is well documented, and lately theorists such as Allen Forte and David Lewin have begun to find some of the same procedures in music of the Second Viennese School.[14] It is also useful to

[11]I wish to thank Joseph Straus for bringing this particular passage to my attention.

[12]Although the realizations of <2310> in Exx. 19a and 19b are subsets of both the diatonic and the octatonic collections, the larger musical contexts in which they are found suggest diatonic interpretations. In the first four measures of *Agon,* the pc content is {F G A B C D}, which is a subset of the diatonic collection, but not of the octatonic collection, and in mm. 10–13, the pc content is {B C C♯ D E F♯ G♯ A}, which is a superset of the diatonic collection, but not of the octatonic collection.

[13]For two recent analyses of this work that utilizes looser notions of expansion than the one proposed here, see Elliott Antokoletz, *Twentieth-Century Music* (Englewood Cliffs, N. J.: Prentice Hall, 1992), 10–14, and Ethan Haimo, "Atonality, Analysis, and the Intentional Fallacy," *Music Theory Spectrum* 18/2 (1996): 167–199.

[14]Allen Forte, "An Octatonic Essay by Webern: No. 1 of the Six Bagatelles for String Quartet, Op. 9," *Music Theory Spectrum* 16/2 (1994): 171–95; idem, *The Atonal Music of Anton Webern* (New Haven: Yale University Press, 1998); and David Lewin, "Some Notes on *Pierrot Lunaire,*" in *Music Theory in Concept and Practice,* ed. James M. Baker, David W. Beach, and Jonathan W. Bernard (Rochester: University of Rochester Press, 1997), 433–57. To my knowledge, there is no evidence to suggest that Webern or Schoenberg consciously chose to juxtapose different modular spaces in their music. However, theorists who are as interested in perception as in authorial intention may find modular transformations to be a useful tool in approaching this repertoire.

Example 18. Stravinsky, *Agon*, mm. 418–427; interaction between the octatonic set-class segment <2310> and its inversion <1023>

Example 19. MODTRANS relationships between the first two movements of *Agon*

a. First theme of the first movement (mm. 1-5).

b. Second theme of the first movement (mm. 10-13).

c. Opening theme of the second movement (mm. 61-62).

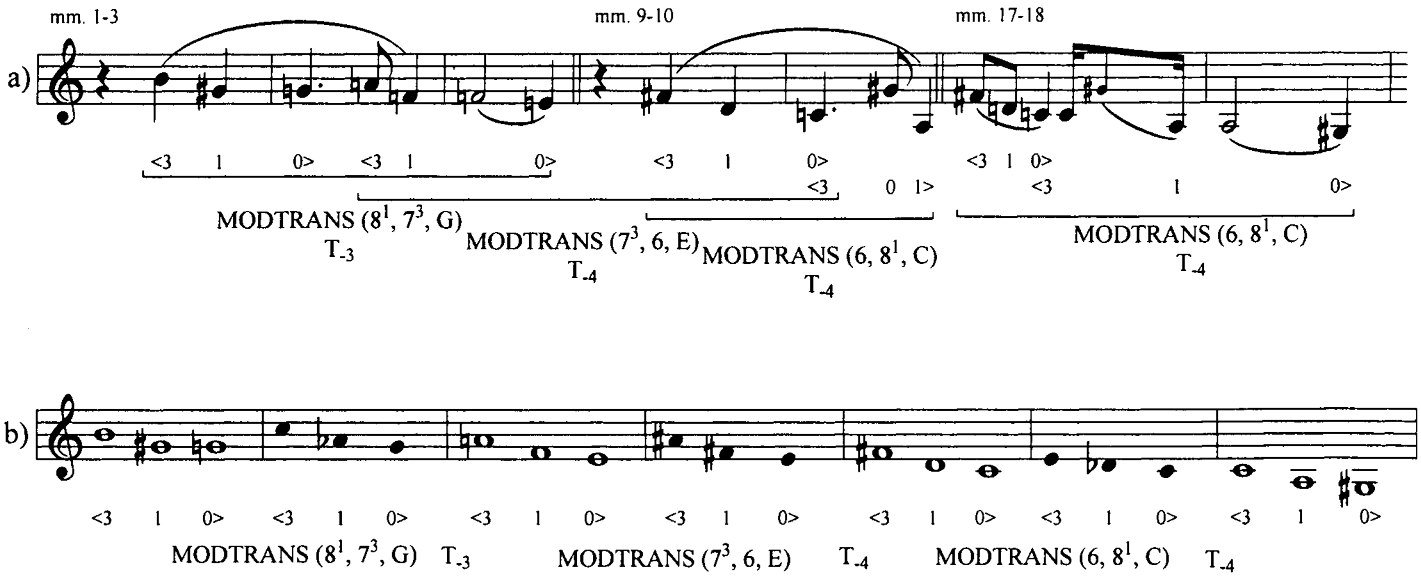
d. MODTRANS operation linking the openings of the first and second movements.

MODTRANS $(7^6, 12, A)$ T_5

e. MODTRANS operation linking the second theme of the first movement to the opening of the second.

MODTRANS $(7^2, 12, B)$ T_3

Example 20. MODTRANS mappings in Schoenberg's Op. 11, No. 1



consider musical perceptions today, which are more attuned to octatonic and whole-tone spaces. Contemporary listeners familiar with music that employs octatonic or whole-tone collections unequivocally may be more likely to perceive evidence of such collections when the source is less overt.

One way in which modular contexts may be suggested is through the use of subsets that limit the number of viable modular interpretations. Though all set classes in a pc set analysis will necessarily be subsets of the chromatic collection, we are not thereby compelled to give preference to a chromatic interpretation, even in musical contexts that are primarily chromatic. Schoenberg's first trichord, (014), is not a subset of the pentatonic, whole-tone, or diatonic collections. This limits the number of viable modular contexts in that trichord may be interpreted to two: octatonic or chromatic. The second trichord, (015), is not a subset of the pentatonic, whole-tone, or octatonic collections, thus limiting the modular context to diatonic or chromatic.

Chromatic interpretations of these trichords do not help us to further understand the relationship between them. However, by interpreting the first as octatonic and the second as diatonic, the two may be related via MODTRANS (8^1, 7^3, G) and T_{-3}, as shown in Example 20b.[15] Suggesting that there is a small-scale motion from octatonic to diatonic in the opening two melodic phrases does not

[15] The resultant diatonic space here could also be interpreted as 7^7 (Locrian). 7^3 (Phrygian) was chosen because of its greater familiarity. The particular rotation an analyst chooses is really only significant as far as it specifies one-to-one mappings in a particular MODTRANS operation, and should not be taken to carry any implications of centricity or collectional priority.

entail a similar interpretation of the two vertical trichords that harmonize these phrases in mm. 1–3, {F G♭ B} and {A B♭ D♭}. Indeed, the interpretation of modular spaces suggested by the compound transformation connecting these two trichords, MODTRANS(8^1, 12, F) and T_4, contradicts the interpretation of modular spaces suggested by the melody. The analyst should resist the temptation to resolve these kind of conflicts, because they may be as essential to an understanding of the music as two conflicting metrical interpretations are to an understanding of a hemiola. Those who are skeptical that two modular spaces may be suggested at once should be able to demonstrate the phenomenon for themselves at the piano by playing an octatonic scale in one hand and a whole-tone scale in the other simultaneously.

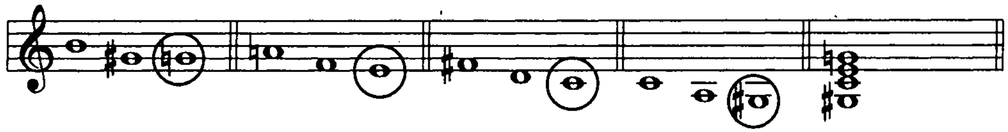
Following the initial mapping of <B G♯ G> onto <A F E>, the latter then maps to the first three notes of the phrase beginning in m. 9, <F♯ D C>, via MODTRANS (7^3, 6, E) and T_{-4}. Again the melodic excerpt is given in Example 20a, its MODTRANS interpretation in Example 20b. This new trichord is also linked by common-tone C to another version of the motive, <C G♯ A> in m. 10, which is (reordered) T_{11} of the <B G♯ G> of m. 1. In the third version of the phrase, mm. 17–18, the trichords of the second phrase are reiterated. The G♯4 in m. 17 is interpreted here as a kind of ornament, an early, octave-displaced arrival of the pitch class that will complete and restore order to the motive in m. 18.[16] The trichords here and in mm. 9–10 are linked via MODTRANS (6, 8^1, C) and T_{-4}, as shown in Example 20b.

Two observations may be made upon examining the four forms of the motive given in Example 20b and the MODTRANS operations that connect them. The first is that the MODTRANS operations might help to define the relationship between the first phrase in mm. 1–3 and the second phrase in mm. 9–10, repeated in mm. 17–18, as something analogous to antecedent and consequent. The MODTRANS operation that binds together the two motive forms of the first phrase moves from an octatonic space to a diatonic space, while the MODTRANS operation that binds together the two forms in the second phrase moves from whole-tone back to octatonic. This motion away from octatonic and back again is analogous to the motion typically associated with antecedent-consequent phrase pairs, in which a move away from some point of stability in the former phrase (be it tonic harmony, or simply a low point in the melodic contour) is answered by a return to that point in the latter.

Example 21 illustrates a second observation. The lowest note in each of the four motive forms is part of a transformational path that outlines the harmony {G♯ C E G}, a tetrachord whose set class, (0148), plays a significant role in the rest of the movement. Example 22 shows three passages in the piece where (0148) is particularly prominent. In mm. 14–17, this tetrachord receives special emphasis as a kind of "magic chord" because Schoenberg gives special instructions to hold down the notes {F A C♯ E} without playing them, so as to provide additional resonance to the same pitch classes being played two octaves below in the left hand. This is the only harmony in the movement to receive such coloristic treatment. This "magic chord," is not only a member of the same set class as the one outlined by the transformational path mentioned above, but is literally the same harmony transposed in pitch space down a minor third, the same interval of transposition that links the first two notes of the first motive form, and of course, the first two notes of the piece. The same set class also constitutes the first four notes of the five-note accompanimental line, <D F♯ A A♯ B>, that appears three times: in mm. 4–5, in m. 6, and in m. 8. This line returns at T_1 in mm. 20–21 and 22–23, and soon after returns inverted as part of a three-voice canon in mm. 25–27. Finally, the tetrachord occurs linearly in m. 28 as <C E♭ G B> and vertically in m. 58 as {E♭ G♭ B♭ D}. If the analyst were unable to perceive the motive forms as relatable by an operation such as MODTRANS, the transformational path carved out

[16]The idea of interpreting some pitches on the surface of Schoenberg's music as ornamental has been argued persuasively by Jack Boss, "Schoenberg on Ornamentation and Structural Levels," *Journal of Music Theory* 38/2 (1994): 187–216.

Example 21. Transformational path of (0148) in Schoenberg's Op. 11, No. 1



Example 22. Motivic use of (0148) in Schoenberg's Op. 11, No. 1

by the transposition levels of these motive forms would consequently be obscured.

II. MODULAR SETS AND MODULAR SET TYPES

We now use the MODTRANS operation to create new equivalence classes. To start, let us define a *modular set* as an unordered collection of step classes. A modular set may be realized in any kind of modular space that comprises all of its step classes; for example, {01346} may be represented in a mod7^1 space as {C D F G B}, or in a mod8^1 space as {C C♯ E F♯ A}, but not in a mod6 space, because that space does not include a step-class 6.

A *modular set-type* (M-type) is defined as an equivalence class including all modular sets that map onto one another under mod12 transposition or inversion. We use mod12 even though the modular set can be represented in any kind of modular space, because the equal-tempered harmonic system (the mod12 system) is the background against which all of the other systems are set into relief. Thus the list of M-types duplicates exactly the conventional listing of mod12 set classes. And yet an M-type typically encompasses much more than a conventional set class, since its members are relatable not only via transposition and inversion but also MODTRANS. For example, the M-type (013) includes not only all members of the conventional set class (013), but also all members of set classes (014), (015), (025), and (026): (013) maps onto (014) under MODTRANS (12, 8^1, 0); it maps onto (015) under MODTRANS (12, 7^3, 0) or MODTRANS (12, 7^7, 0); it maps onto (025) under MODTRANS (12, 8^2, 0), MODTRANS (12, 7^1, 0),

MODTRANS (12, 7^2, 0), MODTRANS (12, 7^5, 0), or MODTRANS (12, 7^6, 0); and it maps onto (026) under MODTRANS (12, 7^4, 0) or MODTRANS (12, 6, 0).

Many M-types represent multiple set classes. An appendix to this article shows the set classes derivable for each of the M-types of cardinalities 3–6, excluding those set classes produced by modular wraparound (MODWRAP). The table displays, among other things, symmetries within each modular space. In the "mod8" column of the entries for trichords (013), (014), (015), and (016), for example, we might symmetrically associate (013) and (016), both of which may be mapped onto (014) or (025), and also associate (014) and (015), both of which may be mapped onto (016) or (026).[17] This is because the M-types (013) and (016) are related by mod8 inversion, as are the M-types (014) and (015). Similarly, in the "mod7" column of the entries for the first five trichords, we might symmetrically associate (012) and (016), both of which may be mapped onto (013) or (024), and also associate (013) and (015), both of which may be mapped onto (015), (025), or (026).[18] This is because the M-types (012) and (016) are related by mod7 inversion, as are the M-types (013) and (015). (014), the center of this symmetry, is inversionally self-symmetrical, mod7. Such symmetrical arrangements occur throughout the table.

Now let us turn to another well-known atonal work and situate its structure within the larger context provided by modular sets and set types. The third of Webern's Five Movements for String Quartet, op. 5 may be viewed as an interplay between (013), (014), (015), and (026), all forms of M-type (013). Example 23 provides an annotated score of this movement, with various forms of these set classes highlighted.

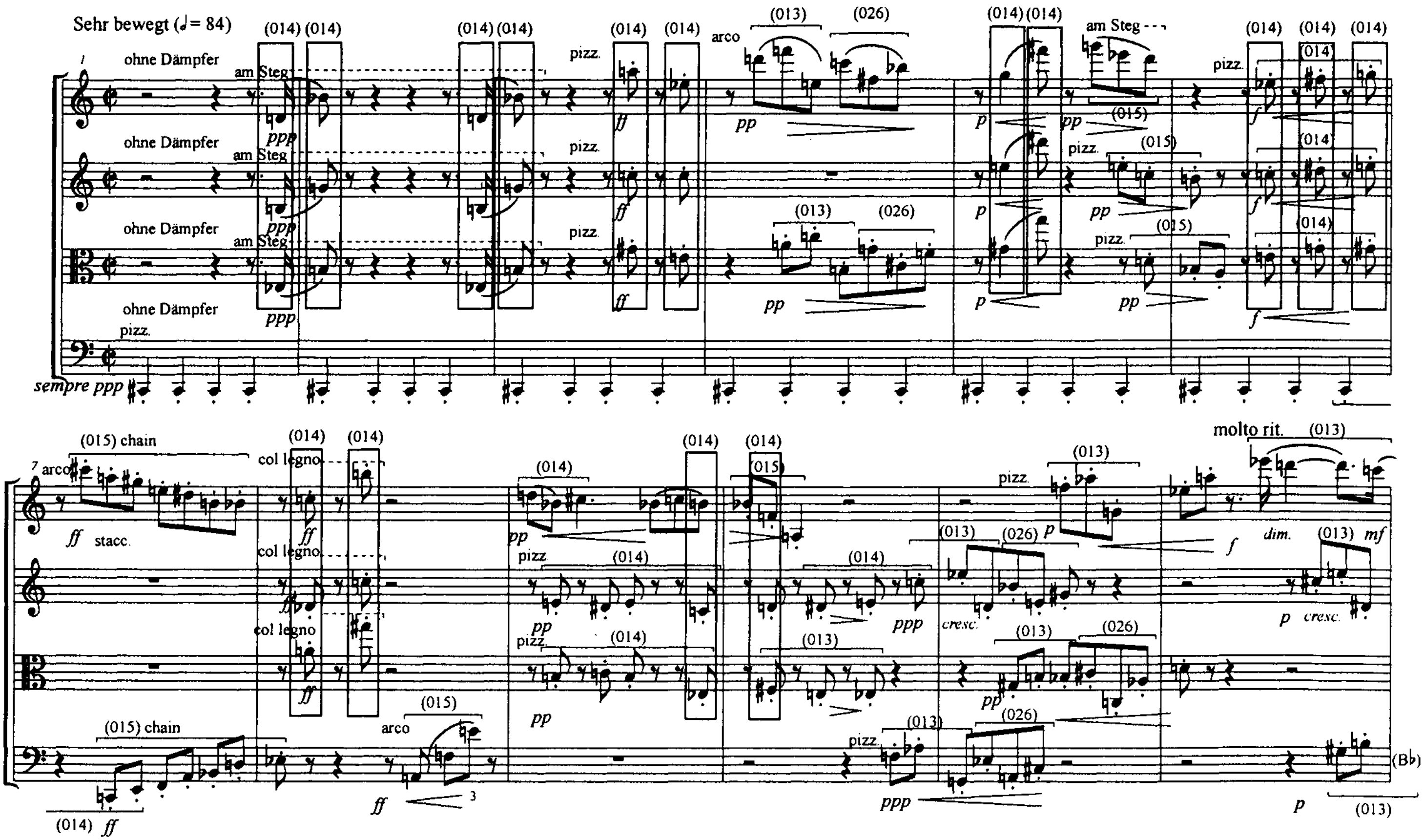
Example 24 provides an analysis of the first theme, which is first presented canonically by the violin I and viola in m. 4, returns in four-part canon in mm. 10–12, continues as a counterpoint to the third theme in mm. 12–13, and reappears in mm. 18–21 as an ostinato figure in the violin II and viola parts. The initial presentation of the theme, given in Example 24a, consists of two three-note subphrases, <D F E> and <C F♯ B♭>, members of set classes (013) and (026) respectively. Let us view the first trichord in a diatonic space, the second in whole-tone; then they are related by MODTRANS (12, 6, D) and T_4, as Example 24b illustrates. Other interpretations are possible. For example, the second trichord could be a diatonic (013) rather than whole tone; a whole-tone representation has been chosen here based on the fact that set class (026) is imbricated twelve times in the whole-tone collection and is thus more indicative of that collection than of the diatonic one, in which it is imbricated only once. Other alternatives include viewing <D F E> as M-type (012) in a diatonic or octatonic space, and <C F♯ B♭> as M-type (014) in octatonic. But these interpretations do not relate the trichords to one another or to much of the material that follows effectively. Interpreting them both as forms of M-type (013) includes them in an all-encompassing view of the piece that better captures its overall organic unity.

Example 25a shows the second theme as it is first presented by the violin I in mm. 9–10; it returns transposed in mm. 22–23 to close the movement. The theme consists of three three-note subphrases, <D B♭ C♯>, <B♭ C♮ B♮>, and <B♭ F A>, the first and third of which are related by MODTRANS (8^1, 7^1, B♭) and T_7, as shown in Example 25b. (The second trichord relates to neither the first nor the third via MODTRANS.) The three-note cello ostinato in mm. 15–21, <B G A♯> is a T_9 transposition of the second theme's first trichord.

[17](013) maps onto (014) via MODTRANS (12, 8^1, 0) or onto (025) via MODTRANS (12, 8^2, 0); (016) maps onto (014) via MODTRANS (12, 8^1, 0) or onto (025) via MODTRANS (12, 8^2, 0). Similarly, (014) maps onto (016) via MODTRANS (12, 8^1, 0) or onto (026) via MODTRANS (12, 8^2, 0); (015) maps onto (016) via MODTRANS (12, 8^1, 0) or onto (026) via MODTRANS (12, 8^2, 0).

[18](012) maps onto (013) via MODTRANS (12, $7^{2, 3, 6, \text{ or } 7}$, 0) or onto (024) via MODTRANS (12, $7^{1, 4, \text{ or } 5}$, 0); (016) maps onto (013) via MODTRANS (12, $7^{1, 3, 4, \text{ or } 7}$, 0) or onto (024) via MODTRANS (12, $7^{2, 5, \text{ or } 6}$, 0). Similarly, (013) maps onto (015) via MODTRANS (12, $7^{3 \text{ or } 7}$, 0) or onto (025) via MODTRANS (12, $7^{1, 2, 5, \text{ or } 6}$, 0) or onto (026) via MODTRANS (12, 7^4, 0); (015) maps onto (015) via MODTRANS (12, $7^{3 \text{ or } 7}$, 0) or onto (025) via MODTRANS (12, $7^{1, 2, 4, \text{ or } 7}$, 0) or onto (026) via MODTRANS (12, 7^6, 0).

Example 23. Annotated score of Webern, Op. 5, No. 3 highlighting set class members of the M-type (013): (013), (014), (015), and (026)



Example 26a provides the third theme, which is first presented in mm. 12–14, and returns, lengthened through internal repetition, in mm. 17–21. It can be partitioned into four successive trichords: <E♭ D C>, <C♯ B G>, <E♭ A F>, and <E F♯ G♭, members of set classes (013), (026), (026), and (013) respectively. The first two are related by MODTRANS (12, 6, C) and T_7, as shown in Example 26b, while the last two are related by MODTRANS (6, 12, E♭) and I_{10}, as shown in Example 26c.

M-type (013) governs the structure of op. 5/3 as it does a great number of other pieces or movements from the atonal repertoire. The same could be said of the first movement of Stravinsky's Concerto in D. As in virtually all of Stravinsky's neoclassical

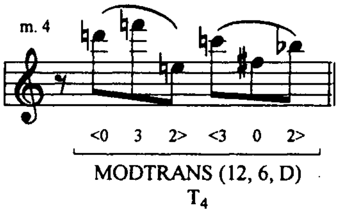
Example 23 [*continued*]

works, the Concerto oscillates between diatonic and octatonic collections.[19] These collections are expressed through the M-type (013), or by one of its supersets, especially (0124), (01245), (0134), (0135), and (01235). Among these, all but (0134) contain (024) as a subset as well; in three of these sets, (024) is imbricated

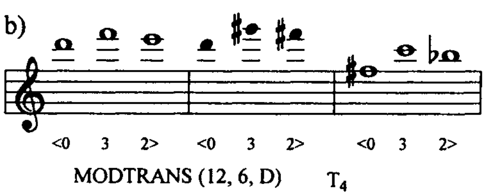
[19]Arthur Berger and Pieter Van den Toorn have both made strong cases for understanding Stravinsky's music in terms of the interaction between diatonic and octatonic collections. See Arthur Berger, "Problems of Pitch Organization in Stravinsky," *Perspectives of New Music* 2/1 (1963): 11–42; and Pieter van den Toorn, *The Music of Igor Stravinsky* (New Haven: Yale University Press, 1983).

Example 24. Webern, Op. 5, No. 3, first theme; transformation from chromatic to whole-tone

a)

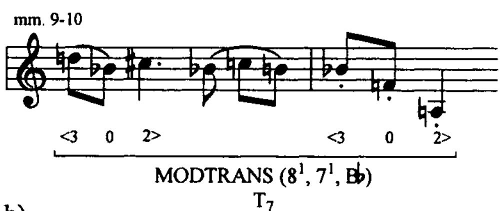


b)

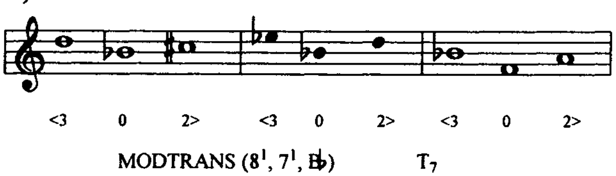


Example 25. Webern, Op. 5, No. 3, second theme; transformation from octatonic to diatonic

a)

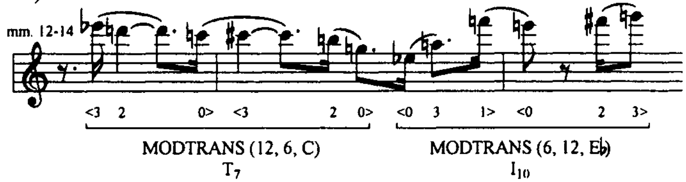


b)

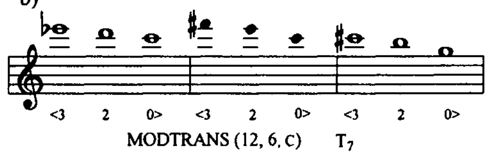


Example 26. Webern, Op. 5, No. 3, third theme; transformation from chromatic to whole-tone, and from whole-tone to chromatic

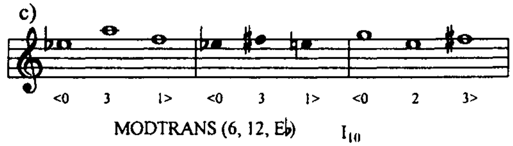
a)



b)



c)



twice. The imbrication of (024) in these larger sets makes possible a number of the tonal allusions with which Stravinsky endowed this work, because (024) is represented by some quality of triad in both diatonic and octatonic modular contexts.

An annotated short score for the introduction and first theme of the Concerto (mm. 1–32) is given as Example 27. A pc-set analysis of the music is represented on the staff at the bottom of each system, under which the M-type and set class of each pc set are listed. Each M-type is followed by the superscripted numbers 7, 8, or 7/8; these indicate the modular space in which the M-type is realized (in the case of 7/8, the set could be interpreted as part of a diatonic or an octatonic space). Several of the movement's essential characteristics are featured in the first thirty-two measures: 1) the M-type (013) recurs frequently, becoming a kind of *Grundgestalt* for the movement; 2) an interpretive conflict is established between the structural F that is a part of the introduction's prevailing octatonic collection, and the recurrence of that same pitch class, spelled as E♯, in the theme, where it must be reinterpreted as a chromatic lower neighbor in D major; 3) a larger harmonic conflict is established between diatonic and octatonic collections through the use of the pc sets (014) and (015), each of which is a subset of only *one* of the larger collections; and 4) a harmonic ambiguity is suggested by the prominence of (025), which is a subset of both diatonic and octatonic collections.

The avenues that further investigations into modular spaces might take are limited only by our imaginations. Besides the music discussed here, many other works by various composers can also be better understood in terms of modular spaces and the interactions between them. An understanding of modular spaces may also help us to move past the foreground of a piece to explore motivic relationships at deeper levels of structure, an issue I have addressed elsewhere.[20] We might further broaden the concept and apply it to other musical parameters. The notion of meter as a modular space, for example, has already been explored compositionally by Milton Babbitt and formalized by David Lewin and Robert Morris.[21] Or one might conceptualize musical texture as a modular space by making the cardinality of the modulus equal to the number of rhythmically independent parts, those parts being organized from 0 to *n* based on the amount of rhythmic activity in each. Such new conceptions, and the concomitant possibilities for interactions within and among different musical dimensions, suggest future directions that are replete with inviting possibilities.

ABSTRACT

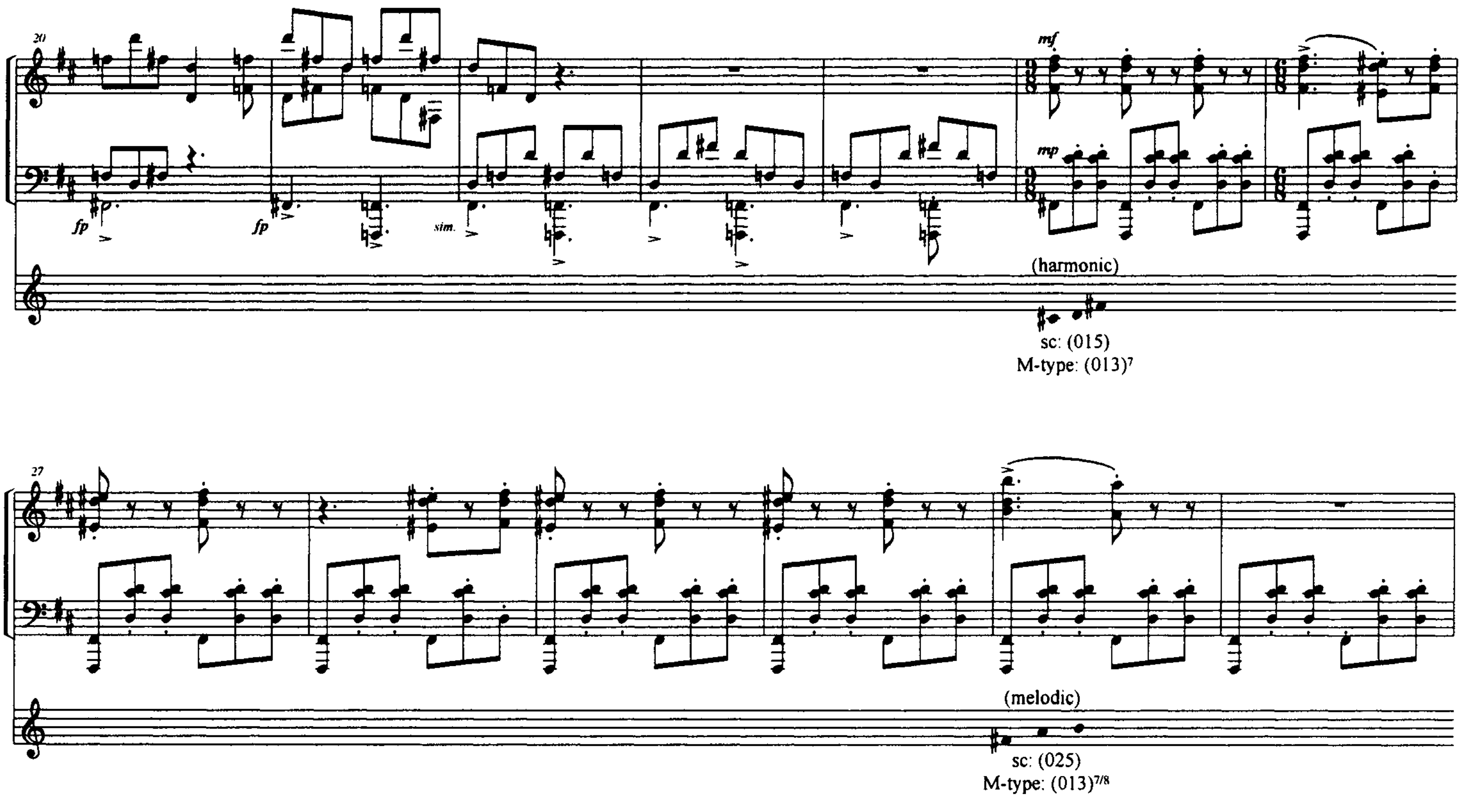
In a lecture at Harvard University in 1943, Bartók acknowledged his discovery and use of a transformation that maps musical entities back and forth between diatonic and chromatic modular systems. But Bartók's transformation need not be limited to these; one can find examples of mappings to and from other modular spaces in Bartók's own music. This article formalizes Bartók's transformation, as well as generalizes it to map musical entities to and from any one of five different spaces: chromatic (mod12), octatonic (mod8), diatonic (mod7), whole-tone (mod6), and pentatonic (mod5). The article then demonstrates that the generalized operation is not linked to any one compositional style in the twentieth century by showing its use in works by Debussy, Stravinsky, and Schoenberg. Finally, it defines a new equivalence class, the modular set type, which groups together those set classes that may be connected via the generalized transformation, and uses the new equivalence class and generalized transformation in analyses of Webern's op. 5, no. 3 and Stravinsky's Concerto in D.

[20]Matthew Santa, "Studies in Post-Tonal Diatonicism: A Mod7 Perspective" (Ph.D. diss., City University of New York, 1999).

[21]For a detailed description of Milton Babbitt's most famous treatment of meter as a modular space see Christopher Wintle, "Milton Babbitt's *Semi-Simple Variations*," *Perspectives of New Music* 14/2–15/1 (1976): 111–54; for Lewin's formalization see David Lewin, *Generalized Musical Intervals and Transformations* (New Haven: Yale University Press, 1987), 23–30; for Morris's formalization see Robert D. Morris, *Composition with Pitch-Classes: A Theory of Compositional Design* (New Haven: Yale University Press, 1987), 301–5.

Example 27. Stravinsky, Concerto in D, analysis of mm. 1–32

Example 27 [*continued*]



Appendix: Set Classes Represented by Each of the M-Types of Cardinalities 3–6

M-type	mod12	mod8	mod7	mod6	mod5
(012)	(012)	(013)	(013), (024)	(024)	(024), (025)
(013)	(013)	(014), (025)	(0125), (025), (026)	(026)	(027), (037)
(014)	(014)	(016), (026)	(016), (027)	(026)	(024), (025)
(015)	(015)	(016), (026)	(015), (025), (026)	(024)	none
(016)	(016)	(014), (025)	(013), (024)	none	none
(024)	(024)	(036)	(036), (037)	(048)	(027), (037)
(025)	(025)	(037)	(036), (037)	(026)	none
(026)	(026)	(036)	(015), (025), (026)	none	none
(027)	(027)	(014), (025)	none	none	none
(036)	(036)	(037)	(016), (027)	none	none
(037)	(037)	(016), (026)	none	none	none
(048)	(048)	none	none	none	none
(0123)	(0123)	(0134), (0235)	(0135), (0235), (0246)	(0246)	(0247), (0257), (0358)
(0124)	(0124)	(0136), (0236)	(0136), (0137), (0237), (0247)	(0248)	(0247), (0257), (0358)
(0125)	(0125)	(0137)	(0136), (0137), (0237), (0247)	(0246)	none
(0126)	(0126)	(0136), (0236)	(0135), (0235), (0246)	none	none
(0127)	(0127)	(0134), (0235)	none	none	none
(0134)	(0134)	(0146)	(0156), (0157), (0257)	(0268)	(0247), (0257), (0358)
(0135)	(0135)	(0147), (0258)	(0158), (0258), (0358)	(0248)	none
(0136)	(0136)	(0347), (0358)	(0136), (0137), (0237), (0247)	none	none
(0137)	(0137)	(0136), (0236)	none	none	none
(0145)	(0145)	(0167), (0268)	(0156), (0157), (0257)	(0123)	none
(0146)	(0146)	(0147), (0258)	(0136), (0137), (0237), (0247)	none	none
(0147)	(0147)	(0137)	none	none	none
(0148)	(0148)	none	none	none	none
(0156)	(0156)	(0146)	(0135), (0235), (0246)	none	none
(0156)	(0156)	(0146)	(0135), (0235), (0246)	none	none
(0157)	(0157)	(0136), (0236)	none	none	none
(0158)	(0158)	none	none	none	none
(0167)	(0167)	(0123)	none	none	none
(0235)	(0235)	(0347), (0358)	(0158), (0258), (0358)	(0268)	none

M-type	mod12	mod8	mod7	mod6	mod5
(0236)	(0236)	(0147), (0258)	(0156), (0157), (0257)	none	none
(0237)	(0237)	(0146)	none	none	none
(0246)	(0246)	(0369)	(0158), (0258), (0358)	none	none
(0247)	(0247)	(0347), (0358)	none	none	none
(0248)	(0248)	none	none	none	none
(0257)	(0257)	(0347), (0358)	none	none	none
(0258)	(0258)	none	none	none	none
(0268)	(0268)	none	none	none	none
(0347)	(0347)	(0167), (0268)	none	none	none
(0358)	(0358)	none	none	none	none
(0369)	(0369)	none	none	none	none
(01234)	(01234)	(01346)	(01356), (01357), (02357)	(02468)	(02479)
(01235)	(01235)	(01347), (02358)	(01358), (02358), (02469)	(02468)	none
(01236)	(01236)	(01347), (02358)	(01356), (01357), (02357)	none	none
(01237)	(01237)	(01346)	none	none	none
(01245)	(01245)	(01367), (02368)	(01368), (01378), (02479)	none	none
(01246)	(01246)	(01369)	(01358), (02358), (02469)	none	none
(01247)	(01247)	(01347), (02358)	none	none	none
(01248)	(01248)	none	none	none	none
(01256)	(01256)	(01367), (02368)	(01356), (01357), (02357)	none	none
(01257)	(01257)	(01347), (02358)	none	none	none
(01258)	(01258)	none	none	none	none
(01267)	(01267)	(01346)	none	none	none
(01268)	(01268)	none	none	none	none
(01346)	(01346)	(01469)	(01368), (01378), (02479)	none	none
(01347)	(01347)	(01367), (02368)	none	none	none
(01348)	(01348)	none	none	none	none
(01356)	(01356)	(01469)	(01358), (02358), (02469)	none	none
(01357)	(01357)	(01369)	none	none	none
(01358)	(01358)	none	none	none	none
(01367)	(01367)	(01347), (02358)	none	none	none
(01368)	(01368)	none	none	none	none
(01369)	(01369)	none	none	none	none
(01457)	(01457)	(01367), (02368)	none	none	none

Appendix *continued*

M-type	mod12	mod8	mod7	mod6	mod5
(01458)	(01458)	none	none	none	none
(01468)	(01468)	none	none	none	none
(01469)	(01469)	none	none	none	none
(01478)	(01478)	none	none	none	none
(01568)	(01568)	none	none	none	none
(02346)	(02346)	(01369)	(01368), (01378), (02479)	none	none
(02347)	(02347)	(01367), (02368)	none	none	none
(02357)	(02357)	(01469)	none	none	none
(02358)	(02358)	none	none	none	none
(02368)	(02368)	none	none	none	none
(02458)	(02458)	none	none	none	none
(02468)	(02468)	none	none	none	none
(02469)	(02469)	none	none	none	none
(02479)	(02479)	none	none	none	none
(03458)	(03458)	none	none	none	none
(012345)	(012345)	(013467), (023568)	(013568), (013578), (023579), (024579)	(02468T)	none
(012346)	(012346)	(013469), (023569)	(013568), (013578), (023579), (024579)	none	none
(012347)	(012347)	(013467), (023568)	none	none	none
(012348)	(012348)	none	none	none	none
(012356)	(012356)	(013479), (014679)	(013568), (013578), (023579), (024579)	none	none
(012357)	(012357)	(013469), (023569)	none	none	none
(012358)	(012358)	none	none	none	none
(012367)	(012367)	(013467), (023568)	none	none	none
(012368)	(012368)	none	none	none	none
(012369)	(012369)	none	none	none	none
(012378)	(012378)	none	none	none	none
(012456)	(012456)	(013679)	(013568), (013578), (023579), (024579)	none	none
(012457)	(012457)	(013479), (014679)	none	none	none
(012458)	(012458)	none	none	none	none
(012467)	(012467)	(013469), (023569)	none	none	none
(012468)	(012468)	none	none	none	none
(012469)	(012469)	none	none	none	none
(012478)	(012478)	none	none	none	none

M-type	mod12	mod8	mod7	mod6	mod5
(012479)	(012479)	none	none	none	none
(012567)	(012567)	(013467), (023568)	none	none	none
(012568)	(012568)	none	none	none	none
(012569)	(012569)	none	none	none	none
(012578)	(012578)	none	none	none	none
(012579)	(012579)	none	none	none	none
(012678)	(012678)	none	none	none	none
(013457)	(013457)	(013679)	none	none	none
(013458)	(013458)	none	none	none	none
(013467)	(013467)	(013479), (014679)	none	none	none
(013468)	(013468)	none	none	none	none
(013469)	(013469)	none	none	none	none
(013478)	(013478)	none	none	none	none
(013479)	(013479)	none	none	none	none
(013568)	(013568)	none	none	none	none
(013569)	(013569)	none	none	none	none
(013578)	(013578)	none	none	none	none
(013579)	(013579)	none	none	none	none
(013679)	(013679)	none	none	none	none
(014568)	(014568)	none	none	none	none
(014579)	(014579)	none	none	none	none
(014589)	(014589)	none	none	none	none
(014679)	(014679)	none	none	none	none
(023457)	(023457)	(013479), (014679)	none	none	none
(023458)	(023458)	none	none	none	none
(023468)	(023468)	none	none	none	none
(023469)	(023469)	none	none	none	none
(023568)	(023568)	none	none	none	none
(023579)	(023579)	none	none	none	none
(024579)	(024579)	none	none	none	none
(02468T)	(02468T)	none	none	none	none